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# Algebra, Analysis and Probability for a Coupled System of Reaction-Diffusion Equations

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*Phil. Trans. R. Soc. Lond. A* 1995 **350**, 69-112
doi: 10.1098/rsta.1995.0003

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This paper is designed to interest analysts and probabilists in the methods of the 'other' field applied to a problem important in biology and in other contexts. It does not strive for generality. After § 1 *a*, it concentrates on the simplest case of a coupled reaction-diffusion equation. It provides a complete treatment of the *existence, uniqueness, and asymptotic behaviour* of monotone travelling waves to various equilibria, both by differential-equation theory and by probability theory. Each approach raises interesting questions about the other.

The differential-equation treatment makes new use of the *maximum principle* for this type of problem. It suggests a numerical method of solution which yields computer pictures which illustrate the situation very clearly. The probabilistic treatment is careful in its proofs of *martingale* (as opposed to merely local-martingale) properties. A new *change-of-measure* technique is used to obtain the best lower bound on the speed of the monotone travelling wave with Heaviside initial conditions. Waves to different equilibria are shown to be related by Doob $h$-transforms. *Large-deviation theory* provides heuristic links between alternative descriptions of minimum wave speeds, rigorous algebraic proofs of which are provided.

Since the paper was submitted, an alternative method of proving *existence* of monotone travelling waves has been developed by Karpelevich *et al.* (1993). We have extended our results in different directions from theirs (one of which is hinted at in § 1 *a*), and have found the methods used here well equipped for these generalizations. See the Addendum.

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## 1. Introduction and summary

### (*a*) *A motivating example*

This subsection is intended only to indicate heuristically one of the directions in which the topic will be developed.

Someone reading the results in this paper and familiar with the theory of harmonic oscillators might well make the following guess.

Consider a *typed* branching diffusion which evolves as follows. Each particle, once born, lives for ever. Each particle has a *type* which is a real number evolving (independently of the types of other particles and of the positions of all particles) as an Ornstein–Uhlenbeck process associated with the operator $\frac{1}{2}\theta(\mathrm{d}^2/\mathrm{d}y^2 - y\mathrm{d}/\mathrm{d}y)$. Here $\theta$ is a positive parameter analogous to temperature. A particle of type $y$ moves on the real line as a driftless Brownian motion with variance coefficient $ay^2$, where $a$ is a positive constant. Moreover, a particle of type $y$ will, in a small time interval of length $h$, give birth to one child (born with its parent's type $y$ and at its parent's current position) with probability $(ry^2+\rho)h + o(h)$, where $r$ and $\rho$ are positive constants. At time 0, there is one particle of type 0 at position 0. Let $L(t)$ be the position at time $t$ of the leftmost particle. Then the wave speed

$$c(\theta) := -\lim_{t\to\infty} t^{-1}L(t)$$

exists almost surely and satisfies

$$c(\theta)^2 = \infty \text{ if } \theta \leqslant 8r, \qquad c(\theta)^2 = 2a\left\{r+\rho+\frac{2(2r+\rho)^2}{\theta-8r}\right\} \text{ if } \theta > 8r.$$

The study of such models will be our concern in sequels. Here, we consider a much simpler model.

### (b) *The classical case: analysis*

Let $a$ and $r$ be positive constants. The Fisher–Kolmogorov–Petrowski–Piscounov (FKPP) equation:

$$\frac{\partial u}{\partial t} = \tfrac{1}{2}a\frac{\partial^2 u}{\partial x^2} + r(u^2 - u), \tag{1.1}$$

where $u = u(t,x)$ is a function on $[0,\infty)\times\mathbb{R}$, has been extensively studied both by analytic techniques (Fisher 1937; Kolmogorov *et al.* 1937), and by probabilistic methods (Watanabe 1967; McKean 1975, 1976; Bramson 1978, 1983; Biggins 1977, 1979, 1992; Uchiyama 1981, 1982; Freidlin 1985, 1991; Neveu 1987; Chauvin & Rouault 1988, 1990; Elworthy & Zhao 1993; Elworthy *et al.* 1993), to name but a few. The analytic facts of primary interest to probabilists (and therefore for the original motivation) are as follows.

For $c \geqslant \sqrt{2ar}$, equation (1.1) has a monotone travelling-wave solution of wave speed $c$:

$$u(t,x) = w(x - ct),$$

where $w$ is a monotone function on $\mathbb{R}$, increasing from 0 at $-\infty$ to 1 at $\infty$; the function $w$ is unique modulo translations. For $c < \sqrt{2ar}$, there exist travelling-wave solutions of wave speed $c$, but none of these is monotonic. ('Spiralling' solutions to coupled reaction-diffusion equations are of great interest to mathematical biologists since work of Turing (see, for example, Britton 1986; Murray 1989).)

### (c) *A coupled reaction-diffusion system: algebra*

Let $a_1$, $a_2$, $r_1$, $r_2$, $q_1$, $q_2$ be positive constants, fixed throughout. Let $\theta$ be a *positive* parameter, to be thought of as analogous to temperature. We consider a generalized FKPP system of equations,

$$\frac{\partial u}{\partial t} = \tfrac{1}{2}A\frac{\partial^2 u}{\partial x^2} + R(u^2 - u) + \theta Q u, \tag{1.2}$$

where $u$ is a vector-valued function from $[0,\infty)\times\mathbb{R}$ to $\mathbb{R}^2$, and where

$$A := \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \qquad R := \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \qquad Q := \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}.$$

(We use ':=' to mean 'is defined to equal'.) Thus, for example,

$$\frac{\partial u_1(t,x)}{\partial t} = \tfrac{1}{2}a_1\frac{\partial^2 u_1(t,x)}{\partial x^2} + r_1[u_1(t,x)^2 - u_1(t,x)] - \theta q_1 u_1(t,x) + \theta q_1 u_2(t,x).$$

The function $u(t,x) := w(x-ct)$ where $w : \mathbb{R} \to \mathbb{R}^2$, provides a travelling-wave solution of (1.2) if (and only if)

$$\tfrac{1}{2}Aw'' + cw' + R(w^2 - w) + \theta Q w = 0, \tag{1.3}$$

so that, for example,

$$\tfrac{1}{2}a_1 w_1''(x) + cw_1'(x) + r_1[w_1(x)^2 - w_1(x)] - \theta q_1 w_1(x) + \theta q_1 w_2(x) = 0. \tag{1.4}$$





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We switch to a 'phase-space' picture, writing $v$ for the column-vector function with components (in the most illuminating labelling)

$$(v_1, v_2, v_3, v_4) := (w_1, w_2, w_1', w_2'),$$

and regarding (1.3) as a first-order equation

$$\frac{\mathrm{d}v}{\mathrm{d}x} = f(v), \tag{1.5}$$

where, for example,

$$f_1(v) = v_3, \qquad f_3(v) = -2a_1^{-1}[cv_3 + r_1(v_1^2 - v_1) - \theta q_1 v_1 + \theta q_1 v_2].$$

Note that (1.5) will have equilibria where $f(v) = 0$, that is, where $(v_1, v_2, v_3, v_4) = (w_1, w_2, 0, 0)$, where the two-dimensional vector $w$ satisfies

$$R(w^2 - w) + \theta Q w = 0.$$

The '*source point*' $S = (0,0,0,0)$ and the '*target point*' $T = (1,1,0,0)$ will always be equilibria. If $r_1 r_2 \geqslant 4\theta^2 q_1 q_2$, there will also be equilibria at the two points

$$E_\pm = (\tfrac{1}{2} + \theta\rho_1 \pm \sqrt{\Delta}, \tfrac{1}{2} + \theta\rho_2 \mp \sqrt{\Delta}, 0, 0), \tag{1.6}$$

where

$$\rho_i := q_i/r_i \text{ and } \Delta := \tfrac{1}{4} - \theta^2\rho_1\rho_2. \tag{1.7}$$

Our primary concern in this paper is with *monotonic waves from $S$ to $T$*, that is, with solutions of (1.3) for which both $w_1$ and $w_2$ are monotonic increasing functions with $w_i(-\infty) = 0$ and $w_i(+\infty) = 1$. We shall also consider monotonic waves from $S$ to $E_\pm$. We defer study of waves from $E_\pm$ to $T$ to another occasion.

Suppose that $r$ is an equilibrium point of (1.5). To study the behaviour of (1.5) in the neighbourhood of $r$, we write $v(x) - r = z(x)$, and expand (1.5) to first order in $z$ to obtain

$$\frac{\mathrm{d}z}{\mathrm{d}x} = K(r)z, \qquad K_{ij}(r) := \frac{\partial f_i}{\partial v_j} \quad \text{evaluated at } v = r.$$

The matrix $K(r)$ is called the *stability matrix* of equation (1.5) at $r$. The dimension $d_\mathrm{s}(r)$ [respectively $d_\mathrm{u}(r)$] of the *stable* (respectively, *unstable*) *manifold* of (1.5) is the number of eigenvalues of $K(r)$ of negative [respectively, positive] real part, counting algebraic multiplicity (see Coddington & Levinson 1955; Hartman 1982; Carr 1981).

The stability matrix $K(T)$ at $T$ is of particular importance to us. We find that

$$K(T) = K_{c,\theta}(T) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \delta_1(\theta q_1 - r_1) & -\delta_1\theta q_1 & -\delta_1 c & 0 \\ -\delta_2\theta q_2 & \delta_2(\theta q_2 - r_2) & 0 & -\delta_2 c \end{pmatrix}, \tag{1.8}$$

where $\delta_i := 2a_i^{-1}$. In particular, the characteristic polynomial of $K_{c,\theta}(T)$ is $\delta_1\delta_2$ times

$$H(\lambda, c, \theta) := F_1(\lambda, c, \theta)F_2(\lambda, c, \theta) - \theta^2 q_1 q_2, \tag{1.9}$$

where

$$(1.10) \qquad F_i(\lambda, c, \theta) := \begin{vmatrix} \lambda & -1 \\ r_i - \theta q_i & \frac{1}{2}a_i\lambda + c \end{vmatrix} = \tfrac{1}{2}a_i\lambda^2 + c\lambda + r_i - \theta q_i.$$

(1.11) **Lemma.** *If $c > 0$, the dimensions of the various stable and unstable manifolds of the equilibria of equation (1.5) may be described as follows:*

(i) $$d_u(S) = 2;$$

(ii) $$d_s(T) = \begin{cases} 4 & \text{if } \theta(\rho_1 + \rho_2) < 1, \\ 3 & \text{otherwise}; \end{cases}$$

(iii) $$\text{if } \Delta > 0, \text{ then } d_s(E_+) = \begin{cases} 4 & \text{if } \rho_2 > \rho_1 \text{ and } \theta(\rho_1 + \rho_2) > 1, \\ 3 & \text{otherwise.} \end{cases}$$

*For $E_-$, the $\rho_2 > \rho_1$ must be replaced by $\rho_1 > \rho_2$.*

(It should be noted that we later make the convention that the types are labelled so that $\rho_1 \geqslant \rho_2$.)

For every $c$, the fact that $d_u(S) + d_s(T) \geqslant 5$ (for our four-dimensional problem) makes it plausible that there exists a (possibly non-monotonic) travelling wave from $S$ to $T$. When the sum of these dimensions is 6 and at least one travelling wave from $S$ to $T$ exists, it is to be expected that there is a 1-parameter family of travelling waves from $S$ to $T$. Bifurcation diagrams for equilibrium solutions of system (1.3) for the $(w_1, \theta)$ and $(w_2, \theta)$ planes are shown in figure 1.

We now begin to consider the existence of *monotonic* travelling waves from $S$ to $T$. For such a travelling wave with speed $c$ to exist, there must exist a 'monotone' eigenvalue of $K_{c,\theta}(T)$ in the sense now to be described.

(1.12) **Definition.** An eigenvalue $\lambda$ of a real $4 \times 4$ matrix $K$ will be called *stable* (*respectively, unstable*) *monotone* if

(i) $\lambda$ is real and negative (positive), and

(ii) $K$ has an eigenvector $v$ corresponding to $\lambda$ with $v_1$ and $v_2$ of the same sign.

(1.13) **Lemma.** *For fixed $\theta > 0$, there exists a finite positive number $c(\theta)$ such that*

*(i) for $c > c(\theta)$, the matrix $K_{c,\theta}(T)$ has precisely two stable monotone eigenvalues, and*

*(ii) for $c < c(\theta)$, the matrix $K_{c,\theta}(T)$ has no stable monotone eigenvalues.*

*For $c > 0$, we have*

(1.14) *$c > c(\theta)$ if and only if there exists a real negative $\lambda$ such that*

$$F_i(\lambda, c, \theta) < 0 \quad (i = 1, 2) \text{ and } H(\lambda, c, \theta) := (F_1F_2)(\lambda, c, \theta) - \theta^2 q_1 q_2 > 0.$$

*There exists precisely one monotone eigenvalue $\lambda(\theta)$, of $K_{c(\theta),\theta}(T)$ of algebraic multiplicity 2 and geometric multiplicity 1, and at $(\lambda(\theta), c(\theta), \theta)$, we have*

$$(1.15) \qquad F_i < 0 \quad (i = 1, 2), \quad H = 0, \quad H_\lambda := \frac{\partial H}{\partial \lambda} = 0.$$

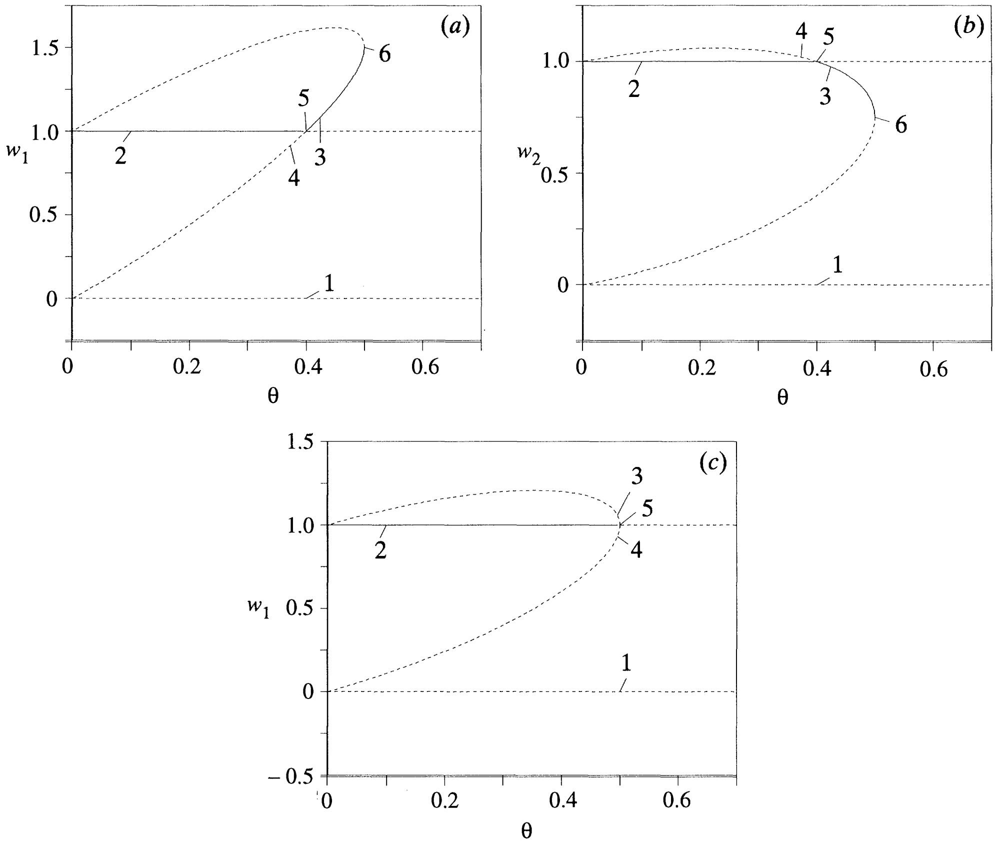


Figure 1. Bifurcation diagrams of equilibria in the $(w_1, \theta)$ and $(w_2, \theta)$ planes, in which the branch labelled 1 is $S$, 2 is $T$ and 3, 4 are $E_\pm$ respectively. Label 5 corresponds to the bifurcation at $\theta = \theta_0$ and 6 to $\theta = \theta^*$. Solid lines represent stable orbits, dashed lines represent unstable ones. (*a*), (*b*) represent the case $\rho_1 > \rho_2$. (*c*) represents the case $\rho_1 = \rho_2$; the bifurcation diagram in the $(w_2, \theta)$ plane is obtained by interchanging the labels 3 and 4. (*a*) $a = (2,1)$, $q = (2,1)$, $r = (1,2)$, $\rho = (2, 0.5)$; (*b*) $a = (2,1)$, $q = (2,1)$, $r = (1,2)$, $\rho = (2, 0.5)$; (*c*) $a = (2,1)$, $q = (2,1)$, $r = (2,1)$, $\rho = (1,1)$.

It will now be convenient to write for a probability measure $m$ on $\{1,2\}$ (so that $0 \leqslant m_1 = 1 - m_2 \leqslant 1$):

(1.16) $$m(a) := m_1 a_1 + m_2 a_2, \qquad m(r) := m_1 r_1 + m_2 r_2, \text{ etc.}$$

Next, we define (the supremum being over all probability measures on $\{1,2\}$, of course)

(1.17) $$c_{\mathrm{F}} := \sup_m \sqrt{2m(a)\,m(r)}, \qquad c_{\mathrm{M}} := \sqrt{2\pi(a)\pi(r)},$$

where $\pi$ is the unique probability measure on $\{1,2\}$ such that $\pi Q = 0$: that is,

(1.18) $$\pi_1 = q_2/(q_1 + q_2), \qquad \pi_2 = q_1/(q_1 + q_2).$$

We use the symbol $c_{\mathrm{F}}$ in deference to Freidlin who did deep work on weak coupling which is here related to the $\theta \downarrow 0$ limit. The suffix '$M$' stands for 'mean', since the chain with $\theta = \infty$ will essentially always be in its invariant measure.

Recall the Perron–Frobenius Theorem. This implies that a square matrix $M$ with all off-diagonal entries strictly positive has a special eigenvalue $\Lambda_{\mathrm{PF}}(M)$ with an associated eigenvector with all entries positive and such that every other eigenvalue of $M$ has real part less than $\Lambda_{\mathrm{PF}}(M)$. Moreover, any eigenvector with all entries positive must be a multiple of the Perron–Frobenius eigenvector (see, for example, Seneta 1981).

(1.19) **Lemma.** (i) *If $c \geqslant c(\theta)$, and $\lambda$ is a stable monotone eigenvalue of $K_{c,\theta}(T)$ with monotone eigenvector $v$, then $-\lambda c$ is the Perron–Frobenius eigenvalue of $\frac{1}{2}\lambda^2 A + \theta Q + R$, and the first two components of $v$ provide the corresponding Perron–Frobenius eigenvector.*

(ii) *For $\theta > 0$,*

$$c(\theta) = \inf_{\mu>0} \Lambda_{\mathrm{PF}}(\tfrac{1}{2}\mu A + \mu^{-1}[R + \theta Q]), \tag{1.20}$$

*the infimum being attained when $\mu = -\lambda(\theta)$.*

(iii) *We have*

$$c(\theta) = \inf_{\mu>0} \sup_{m} \{\tfrac{1}{2}\mu m(a) + \mu^{-1}[m(r) - \theta I(m, Q)]\}, \tag{1.21}$$

*where*

$$I(m, Q) := (\sqrt{m_1 q_1} - \sqrt{m_2 q_2})^2. \tag{1.22}$$

(iv) *The infimum and supremum in (1.21) may be interchanged, so that*

$$c(\theta)^2 = 2 \sup_{m} \{m(a)\,[m(r) - \theta I(m, Q)]\}. \tag{1.23}$$

Note that $I(\pi, Q) = 0$. The agreement of (1.20) and (1.21) is a trivial case of the Legendre transformation which occurs in the celebrated Donsker–Varadhan Theorem on large deviations for occupation times (see Varadhan 1984; Deuschel & Stroock 1989; Ellis 1985). The large-deviation heuristic for (1.23) will be explained in § 1 *f*.

(1.24) **Lemma.** *The function $c(\cdot)$ is non-increasing on $(0, \infty)$, and*

$$\lim_{\theta \downarrow 0} c(\theta) = c_{\mathrm{F}}, \qquad \lim_{\theta \uparrow \infty} c(\theta) = c_{\mathrm{M}}.$$

*The functions $c(\cdot)$ and $\lambda(\cdot)$ are real analytic on $(0, \infty)$.*

*Monotonic waves from $S$ to $E_{\pm}$: a Doob h-transform.* The following result is interesting for many reasons.

(1.25) **Lemma.** *Suppose that $\theta$ is fixed at a value where $\Delta \geqslant 0$, so that $E_+$ and $E_-$ exist. If $E = (\alpha_1, \alpha_2, 0, 0)$ denotes either $E_+$ or $E_-$, then the substitution*

$$\tilde{q}_i := q_i \alpha_j / \alpha_i \quad (j \neq i), \qquad \tilde{r}_i := r_i \alpha_i, \qquad \tilde{u}_i := u_i / \alpha_i, \qquad \tilde{w}_i := w_i / \alpha_i \tag{1.26}$$

*transforms (1.2) and (1.3) into their $\sim$ versions, monotonic waves from $S$ to $E$ for the original problem corresponding exactly to monotonic waves from $S$ to $T$ for the $\sim$ problem. Lemmas 1.13 and 1.19 therefore automatically transfer to the*





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*case when $T$ is replaced by $E_+$ or $E_-$. Let the critical $c$-values associated with $E_+$ and $E_-$ in the untransformed variables be written $c_+(\theta)$ and $c_-(\theta)$.*

Because the transformation (1.26) is $\theta$-dependent, we do of course lose the monotonicity in $\theta$ of the 'minimum wave speed' $c_\pm(\theta)$ from $S$ to $E_\pm$ for the original model.

In the next subsection and in § 3, we examine some differential-equation theory for the travelling-wave system (1.3). In this theory, $E_+$ and $E_-$, when they exist, are treated on exactly the same basis as $T$. In the probability theory, attention is focused initially on waves from $S$ to $T$. However, the transformation (1.26) fits naturally into the probability, and brings the study of waves from $S$ to $E_+$ into its scope, because, as will be explained in § 4, this transformation corresponds to a familiar Doob $h$-transform.

(1.27) **Convention.** *We shall henceforth suppose for definiteness that*

$$\rho_1 \geqslant \rho_2, \text{ where } \rho_i := q_i/r_i \text{ as usual.}$$

*We define*

$$(1.28) \qquad \theta_0 := (\rho_1+\rho_2)^{-1}, \qquad \theta^* := (4\rho_1\rho_2)^{-1/2}, \text{ so that } 0<\theta_0\leqslant\theta^*.$$

Thus $E_\pm$ will exist when $0<\theta\leqslant\theta^*$. Moreover,
when $\theta=0$, then $E_+=(1,0,0,0)$ and $E_-=(0,1,0,0)$,
when $\theta=\theta_0$, then $E_+=(2\rho_1\theta_0, 2\rho_2\theta_0, 0, 0)$ and $E_-=T=(1,1,0,0)$,
when $\theta=\theta^*$, then $E_+=E_-=\frac{1}{2}(1+\sqrt{\rho_1/\rho_2}, 1+\sqrt{\rho_2/\rho_1}, 0, 0)$.
Some relations between the various critical $c$-values are included in the following lemma. We write $c(0)$, etc., for what should be written $c(0+)$, etc.

(1.29) **Lemma.** *The following results hold:*

$$c_+(\theta^*)=c_-(\theta^*), \quad c_-(\theta_0)=c(\theta_0),$$

$$c'(\theta)\leqslant 0 \text{ for all } \theta>0, \quad c'_\pm(\theta)>0 \text{ if } 0<\theta<\tfrac{1}{2}(4\rho_1^2+\rho_1\rho_2)^{-1/2},$$

$$c(0)>\max\{c_+(0),c_-(0)\} \text{ if } \min\{a_1/a_2+r_1/r_2,\ a_2/a_1+r_2/r_1\}>2,$$

$$c(0)=\max\{c_+(0),c_-(0)\} \text{ otherwise.}$$

All results in this § 1 $c$ are proved in § 2.

### (d) The main ODE theorem

We now give our main analytic result on the existence and uniqueness of travelling waves. It is worth noting that its proof is valid for problems with nonlinearities considerably more general than those in the model (1.3).

(1.30) **Theorem.** *Suppose that $\theta>0$ and $c>0$. If $c\geqslant c(\theta)$ (respectively, $c_+(\theta)$, $c_-(\theta)$), then there exists one and, modulo translation, only one monotone solution of (1.3) with*

*$w(x)\to S$ as $x\to-\infty$ and $w(x)\to T$ (respectively, $E_+$, $E_-$) as $x\to+\infty$.*

*In particular, if $\theta\in(0,\theta^*)$, then, for sufficiently large $c$, all three monotone travelling waves exist.*

The proof of this result in § 3 corresponds closely to the numerical analysis described in the next subsection, and gives a very clear picture of the paths. Other approaches to the existence of heteroclinic orbits in related situations are available. See, for example, Dunbar (1984, 1986), and, for a powerful approach based on Leray–Schauder index theory (Vol'pert & Vol'pert 1990).

*A Lyapunov functional.* For completeness, though it plays no part in our proof of the existence of monotone travelling waves, we note that when an inner product of equation (1.3) with diag $(q_2, q_1)w'$ is taken, then a Lyapunov functional which is monotone on all trajectories of (1.3) emerges. This functional is an important ingredient in the treatment of general (not necessarily monotone) heteroclinic orbits connecting the equilibria of (1.3). For monotone orbits, it gives too weak a bound: in particular, Lemma 3.1 does not follow from it.

### (*e*) *Numerical results*

In this subsection we present the outcome of numerical experiments on equation (1.3) that serve to illustrate the algebra and analysis. Loci with $\theta$ of the critical wave speeds $c(\theta)$ and $c_\pm(\theta)$ are computed as are profiles of travelling waves from $S$ to $T$. Also, in figures 3 and 4, the behaviour of trajectories on the unstable manifold of $S$, which is used heavily in the proof of Theorem 1.30 (see § 3), is plainly depicted. We strongly recommend the reader to consult these figures when reading § 3.

Two different numerical techniques have been used. First, direct numerical integration of (1.3) using a variable order, variable step-size Adam's method (NAG routine D02CBF) was used to create the unstable manifold pictures. The initial conditions were taken to be

$$(w(0), w'(0)) := v(0) = \epsilon[(\cos\phi)\underline{w} + (\sin\phi)\overline{w}], \tag{1.31}$$

where, in the notation introduced in § 3 below,

$$\underline{w} := (\underline{v}_1, \underline{v}_2, \underline{\lambda}\,\underline{v}_1, \underline{\lambda}\,\underline{v}_2) \quad \text{and} \quad \overline{w} := (\overline{v}_1, \overline{v}_2, \overline{\lambda}\overline{v}_1, \overline{\lambda}\overline{v}_2)$$

are vectors spanning the unstable eigenspace (linear approximation to the unstable manifold) of $S$, $\epsilon$ is a small positive number and $\phi$ is a parameter that we allow to vary. By standard theorems on unstable manifolds (see, for example, Coddington & Levinson 1955) solutions with initial conditions (1.31), for $\epsilon$ sufficiently small will form good approximations to the unstable manifold of $S$ over a (fixed) finite time interval.

The second numerical technique is the continuation of solutions (either equilibria or heteroclinic orbits) of (1.3) as parameters are varied. Such path-following we have implemented using the code AUTO (Doedel *et al.* 1991*a*, *b*) which incorporate pseudo-arclength continuation and so can compute around limit points. The continuation of equilibria (with possible extra algebraic conditions defining $c(\theta)$ or $c_\pm(\theta)$) is then a standard task, AUTO being able to compute stability and detect bifurcations. To continue paths of heteroclinic orbits from $S$ to $T$, we have used the approach of Beyn (1990); see also Friedman & Doedel (1991), Doedel *et al.* (1991*b*) for a similar method. Here, one approximates the boundary-value problem (1.3) subject to

$$v(x) \to S \quad \text{as} \quad x \to -\infty, \qquad v(x) \to T \quad \text{as} \quad x \to \infty,$$

by a two-point boundary-value problem on a (large) finite interval $[0, \tau]$ with boundary conditions,

$$\mathcal{L}_{\mathrm{s}}(v(0)) = 0, \quad \mathcal{L}_{\mathrm{u}}(v(\tau)) = 0, \tag{1.32}$$

where $\mathcal{L}_{\mathrm{s}}$ and $\mathcal{L}_{\mathrm{u}}$ are projection operators onto subspaces orthogonal to the unstable eigenspace of $S$ and the stable eigenspace of $T$ respectively. The computation of paths of solutions to (1.3) with boundary conditions (1.32) as a parameter is varied can readily be implemented using AUTO. A similar approach can be used to follow heteroclinic orbits from $S$ to $E_{\pm}$ or, indeed, from $E_{\pm}$ to $T$.

First, in figure 1, we illustrate bifurcation diagrams with $\theta$ of the equilibrium solutions of (1.3) for illustrative values of the other parameters $a := (a_1, a_2)$, $q := (q_1, q_2)$ and $r := (r_1, r_2)$. The stability properties of each equilibrium is as stated in Lemma 1.11. Note in particular that $T$ undergoes a transcritical bifurcation at $\theta = \theta_0$, that $E_{\pm}$ coalesce at a saddle-node bifurcation when $\theta = \theta^*$, and that $S$ undergoes no bifurcation. This behaviour is independent of $c$ and qualitatively depends on the parameters only in the relative magnitudes of $\rho_1$ and $\rho_2$.

Next, we illustrate Lemma 1.29 on the nature of the curves $c(\theta)$ and $c_{\pm}(\theta)$ which define the minimum wave speeds for monotone travelling waves from $S$ to $T$ and $E_{\pm}$ respectively. Figure 2 *a–c* depicts these curves for three different sets of values of the parameters $a$, $q$ and $r$. Figure 2 *d* magnifies part of figure 2 *c*. In each of the three cases, it can be observed that $c'(\theta) < 0$ for the complete range of $\theta$ plotted, that $c'_{\pm}(\theta) > 0$ for all sufficiently small $\theta$, $c_+(\theta^*) = c_-(\theta^*)$ and $c(\theta_0) = c_-(\theta_0)$. These observations verify the first part of Lemma 1.29. The parameter values used in ($a$) illustrate the case

$$\frac{a_2}{a_1} + \frac{r_2}{r_1} < 2$$

and it can be seen that $c(0) = \max\{c_+(0), c_-(0)\}$ $(= c_+(0))$ as stated in the second part of Lemma 1.29. Notice further that $c_-$ is a monotonically increasing function of $\theta$ in this case, that $c_+(\theta) > \max\{c(\theta), c_-(\theta)\}$ for all $\theta \in (0, \theta^*)$, and that $c_+$ has a single local maximum in this interval.

Figure 2$b, c$ illustrates the complementary case where

$$\min\left\{\frac{a_2}{a_1} + \frac{r_2}{r_1}, \frac{a_1}{a_2} + \frac{r_1}{r_2}\right\} > 2.$$

In both figures $c(0) > \max\{c_+(0), c_-(0)\}$, as stated by the second part of Lemma 1.29; moreover $c_+(0) = c_-(0)$ in (b). This latter equality is not inconsistent with the conclusions of the lemma and is due to the symmetry,

$$\frac{a_2}{a_1} = \frac{r_1}{r_2} = 2.$$

For this case, as in ($a$), $c_-$ is monotonically increasing and $c_+$ has a single local maximum. For the parameter values used in ($c$), which are a small perturbation of those in ($b$) (the only parameter to change is $a_1$), $c_+(0) < c_-(0)$ and there exists a $\theta \in (0, \theta_0)$ such that $c_+(\theta) = c_-(\theta)$. Notice also with reference to figure 2 *d*, which is a magnification of part of figure 2 *c*, that $c_-$ has developed a local maximum and a local minimum and is such that $c_-(\theta) = c(\theta)$ for three distinct values of $\theta \in (0, \theta_0)$. We conclude that the ordering of $c(\theta)$ and $c_{\pm}(\theta)$, the three minimum

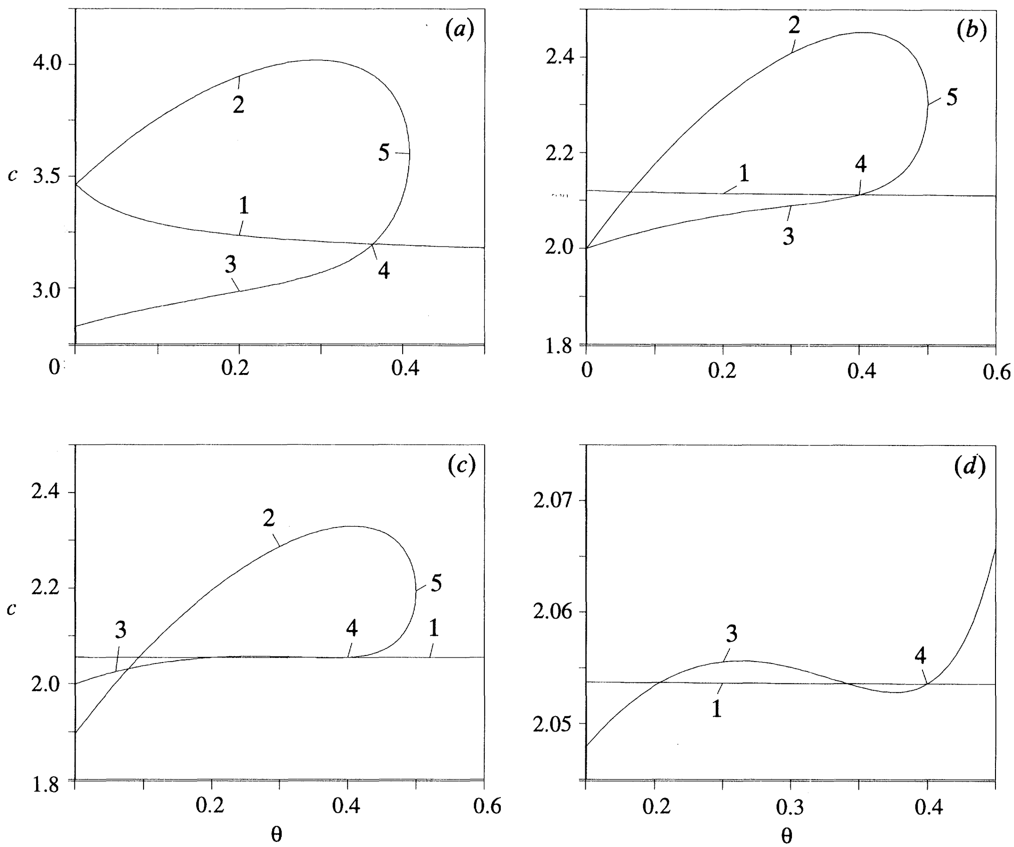


Figure 2. The critical curves $c(\theta)$ (labelled 1), $c_+(\theta)$ (2), and $c_-(\theta)$ (3). Label 4 corresponds to $\theta = \theta_0$ and 5 to $\theta = \theta^*$. (*a*) $a = (2, 1)$, $q = (6, 3)$, $r = (3, 4)$; (*b*) $a = (2, 1)$, $q = (2, 1)$, $r = (1, 2)$; (*c*) $a = (1.8, 1)$, $q = (2, 1)$, $r = (1, 2)$; (*d*) $a = (1.8, 1)$, $q = (2, 1)$, $r = (1, 2)$.

wave speeds for monotone travelling waves to $T$ and $E_\pm$ respectively, depends crucially on the values of the parameters $a$, $q$ and $r$ as well as on $\theta$.

Figures 3 and 4 depict trajectories on a small piece of the unstable manifold of $S$ projected onto the $(w_1, w_2)$-plane for the values of $a$, $q$ and $r$ used in figure 2 *a*. In each figure, orbits in the unstable manifold are represented by dashed lines, solutions corresponding to which travel outwards from $S = (0, 0)$ as $x$ increases. Also the parabolae,

$$P_1 : \quad \theta q_1 w_2 - (r_1 + \theta q_1) w_1 + r_1 w_1^2 = 0,$$
$$P_2 : \quad \theta q_2 w_1 - (r_2 + \theta q_2) w_2 + r_2 w_2^2 = 0,$$

the significance of which is explained in § 3, are superimposed on the trajectories. An intersection point between the two parabolae corresponds to an equilibrium of (1.3).

Figure 3*a*–*d* illustrates, for three different values of $c$, the case where $\theta < \theta_0$ so that there are a total of four equilibria, of which only $T$ is stable ($d_s(T) = 4$). In (*a*), $c > \max\{c_\pm(\theta), c(\theta)\}$ so that Theorem 1.30 states there to be a monotone




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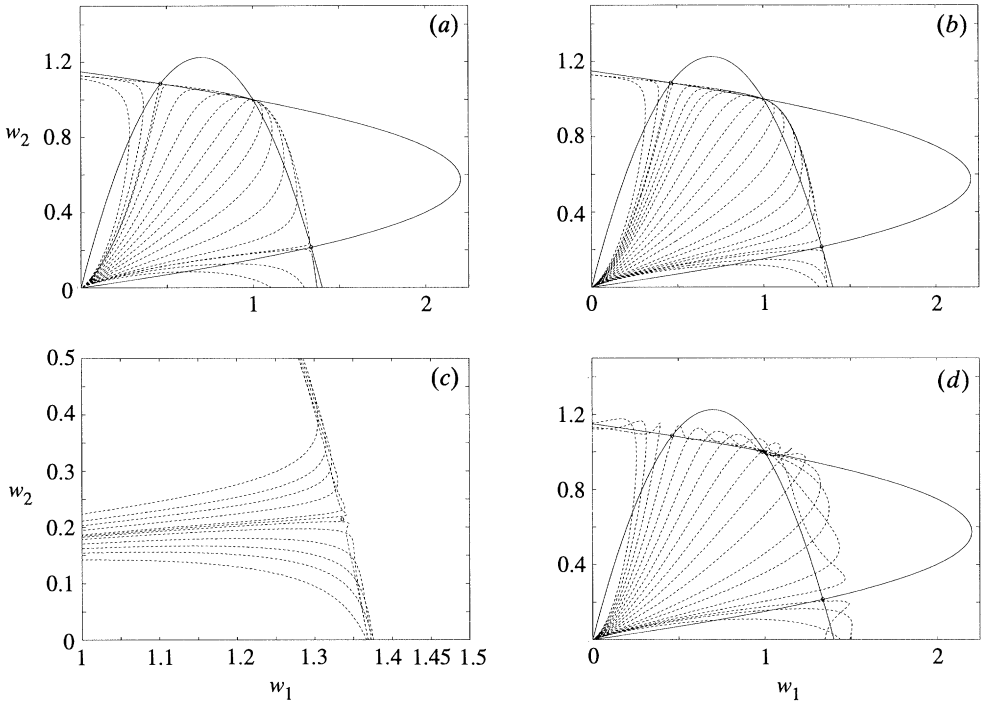


Figure 3. Solutions on the unstable manifold of $S$ for $\theta < \theta_0$ at parameter values for which $c(\theta) = 3.23589$, $c_+(\theta) = 3.94852$ and $c_-(\theta) = 2.98521$. (*a*) $a = (2,1)$, $q = (6,3)$, $r = (3,4)$, $\theta = 0.2$, $c = 4.5$; (*b*) $a = (2,1)$, $q = (6,3)$, $r = (3,4)$, $\theta = 0.2$, $c = 3$; (*c*) $a = (2,1)$, $q = (6,3)$, $r = (3,4)$, $\theta = 0.2$, $c = 3$; (*d*) $a = (2,1)$, $q = (6,3)$, $r = (3,4)$, $\theta = 0.2$, $c = 1.5$.

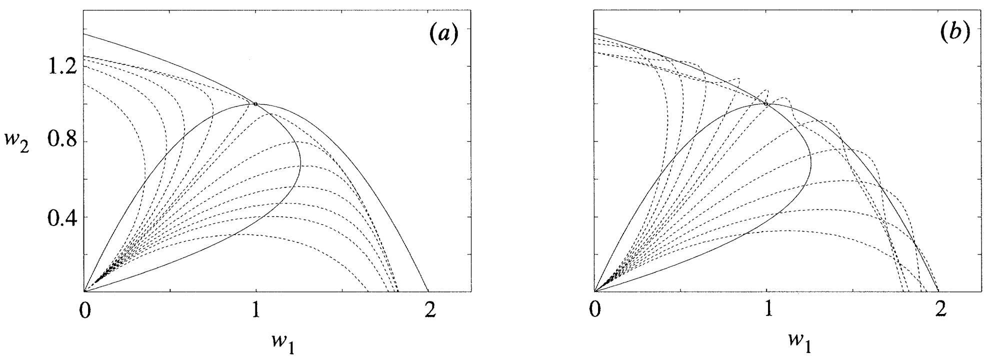


Figure 4. Solutions on the unstable manifold of $S$ for $\theta > \theta^*$ at parameter values for which $c(\theta) = 3.18153$. (*a*) $a = (2,1)$, $q = (6,3)$, $r = (3,4)$, $\theta = 0.5$, $c = 4.5$; (*b*) $a = (2,1)$, $q = (6,3)$, $r = (3,4)$, $\theta = 0.5$, $c = 1.5$.

travelling wave (heteroclinic orbit of (1.3)) connecting $S$ to each of $T$ and $E_\pm$. Moreover, each monotone connection is unique. From the figure we can see that there are a continuum of connections from $S$ to $T$, which contain orbits that approach $T$ tangent to both positive and negative multiples of a non-positive eigenvector. These two kinds of orbit form two separate components of the set of

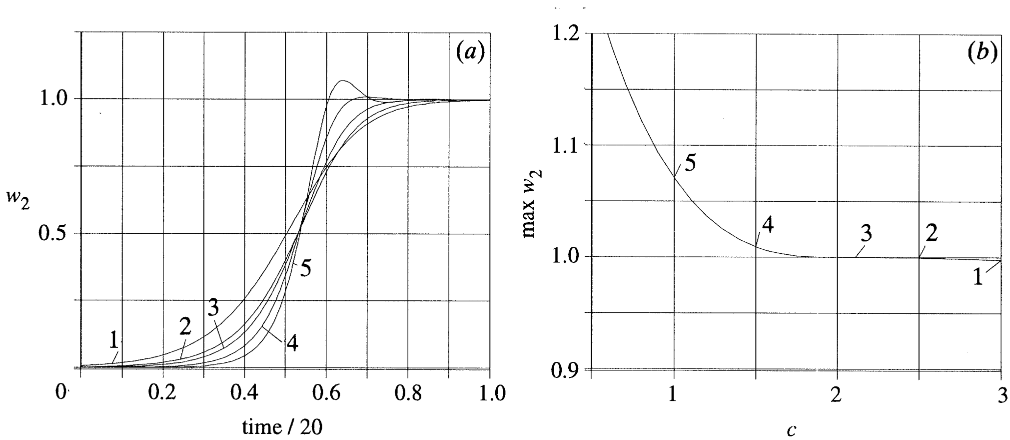


Figure 5. Continuation of travelling waves with $c$ for $a = (2, 1)$, $q = (2, 1)$, $r = (1, 2)$ and $\theta = 1$. ($a$) Profiles of travelling waves, ($b$) maximum $w_2$-value of travelling wave against $c$. ($a$) $a = (2, 1)$, $q = (6, 3)$, $r = (3, 4)$, $\theta = 0.2$, $c = 3$; ($b$) $a = (2, 1)$, $q = (6, 3)$, $r = (3, 4)$, $\theta = 0.2$, $c = 1.5$.

connections to $T$, the boundary between which is the monotone connecting orbit. The outer limiting trajectories of each component form the monotone travelling waves to $E_+$ and $E_-$. In figure 3 $b$, $c_-(\theta) < c < \min\{c_+(\theta), c(\theta)\}$ and so a monotonic travelling wave from $S$ should only exist to $E_-$. Once again a continuum of connections can be observed from $S$ to $T$, however the non-monotonicity of all these connections is not apparent from the figure (observe from figure 2 $a$ that the value $c = 3$ used is only a little smaller than $c(\theta) \approx 3.23589$). The limiting trajectories of this continuum are once again heteroclinic orbits from $S$ to $E_\pm$. By examining figure 3 $c$, which is a magnification of part of figure 3 $b$, it can now be observed that the connection to $E_+$ is not monotonic ($c_+(\theta) \approx 3.94852$). In figure 3 $d$, $c < \min\{c(\theta), c_\pm(\theta)\}$ and so there is no monotone connection from $S$ to any of $T$ or $E_\pm$. A continuum of trajectories from $S$ to $T$ with its limiting trajectories forming connections to $E_\pm$ can once again be observed in the figure, although due to complex eigenvalues at all three target points the trajectories look rather confusing when projected onto a plane.

Figure 4$a, b$ illustrates, for $c > c(\theta)$ and $c < c(\theta)$ respectively, the case where $\theta > \theta^*(\geqslant \theta_0)$ so that $E_\pm$ do not exist and $T$ is unstable. In the former case, a monotone travelling wave from $S$ to $T$ can be inferred from the existence of trajectories in the figure which approach a neighbourhood of $T$ monotonically but leave along different components of a non-positive eigendirection. In figure 4 $b$, the existence of a connection can similarly be inferred but it is clearly non-monotonic.

Figure 5$a, b$ shows the results of following the paths of travelling waves from $S$ to $T$ as the parameter $c$ is decreased. The values of the other parameters are as in figure 2 $a$ with $\theta = 1 > \theta^*$, so that $T$ is unstable, and travelling waves from $S$ to $T$ are consequently isolated. In ($a$) profiles of the $w_2$-coordinate of travelling waves are depicted for various different $c$-values which can be read off from corresponding labels in ($b$). Profiles of the $w_1$-coordinate are similar. Label 2 corresponds to the travelling wave for $c = c(\theta) = 2.11058$. Notice that for $c > c(\theta)$ (e.g. at label 1, where $c = 3$) the profile is monotone, whereas for $c < c(\theta)$ (e.g. at label 5, where $c = 1$) the $w_2$-coordinate can be seen to oscillate around the value 1 at the right-hand endpoint. Figure 5 *b* shows how the maximum value of $w_2$ for the numerically computed travelling wave varies with $c$. A similar figure was obtained by plotting the maximum of $w_1$ against $c$. Recall that, due to the truncation of the infinite interval of the boundary-value problem being solved and to the nature of the right-hand boundary condition (1.32), we should expect the maximum value of $w_1$ or $w_2$ to be slightly less than 1 for a monotone connection from $S$ to $T$. It can be observed from the figure that the maximum value increases through 1 as $c$ is decreased through $c \approx 2$, which is consistent with the minimum speed of monotonic travelling waves being $c = 2.11058$.

### (*f*) *Probability (and analysis)*

We consider a two-type branching Brownian motion. At time $t \geqslant 0$, there are $N(t)$ particles, the $k$th particle – *in order of birth* – having *position* $X_k(t)$ in $\mathbb{R}$ and *type* $Y_k(t)$ in $I := \{1, 2\}$. The *state* of our system at time $t$ is therefore

$$(N(t);\ X_1(t), \ldots, X_{N(t)}(t);\ Y_1(t), \ldots, Y_{N(t)}(t)). \tag{1.33}$$

Particles, once born, behave independently of one another. Each particle lives for ever. The type of a particle (once born) is an autonomous Markov chain on $I$ with $Q$-matrix $\theta Q$. While a particle is of type $y$ (in $I$), its position varies as a Brownian motion on $\mathbb{R}$ of zero drift and constant variance coefficient $a(y)$, and it gives birth – *to one child each time, at its own current position and of its own current type* – in a Poisson process of rate $r(y)$. We write

$$\mathbb{P}_{x,y}, \text{ with associated expectation } \mathbb{E}_{x,y}, \tag{1.34}$$

for the law of our process when it starts from one particle at position $X_1(0) = x$ and of type $Y_1(0) = y$. By martingale (respectively, local martingale, supermartingale,...) we mean a process which is for every $\mathbb{P}_{x,y}$ a martingale (respectively,...) relative to the natural filtration ($\mathbb{P}_{x,y}$-augmented if it makes you happier) of the process at (1.33).

Here are some of our main results. We concentrate on those related to the differential-equation theory, and thereby (to some extent) play down the true role of probability theory which is to study sample paths. Many interesting probabilistic and analytic problems are left to various follow-up papers.

(1.35) *Notation.* In the remainder of this subsection, we shall write

$$u(t, x, y) \quad (y \in I) \text{ rather than } u_y(t, x)$$

in describing a map $u : [0, \infty) \times \mathbb{R} \times I \to \mathbb{R}$, and, similarly,

$$w(x, y) \text{ rather than } w_y(x)$$

in describing a map $w : \mathbb{R} \times I \to \mathbb{R}$. At the same time, let the vector $(v_1, v_2)$ be denoted by $(v(1), v(2))$.

(1.36) **Theorem.** (i) *If $u$ satisfies the reaction-diffusion system* (1.2) *with* $0 \leqslant u \leqslant 1$ *on* $[0, \infty) \times \mathbb{R} \times I$ *and with initial condition*

$$u(0, x, y) = f(x, y), \tag{1.37}$$





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*then $u$ has a McKean representation:*

(1.38) $$u(t,x,y)=\mathbb{E}_{x,y}\prod_{k=1}^{N(t)} f(X_k(t),Y_k(t)).$$

(ii) *If $w$ is a $C^2$ function on $\mathbb{R}\times I$, then*

$$\prod_{k=1}^{N(t)} w(X_k(t)+ct,Y_k(t))$$

*is a local martingale if and only if $w$ solves the travelling-wave system* (1.3), *namely*

$$\tfrac{1}{2}Aw''+cw'+\theta Qw+R(w^2-w)=0.$$

*If $f$ is a $C^2$ function on $\mathbb{R}\times I$, then*

$$\sum_{k=1}^{N(t)} f(X_k(t)+ct,Y_k(t))$$

*is a local martingale if and only if $f$ solves the linearization of* (1.3) *at the point* $(1,1)$*:*

$$\tfrac{1}{2}Af''+cf'+\theta Qf+Rf=0.$$

The second part of the theorem gives a nice interpretation of linearization. It also allows us to take logarithms, turning a product into a sum, and thereby obtaining information about the non-linear system from the linear one. This key idea of McKean's is exploited in the proof of Theorem 1.41. First, we need the relevant information about 'additive' martingales.

(1.39) **Theorem.** *Let $c>c(\theta)$. Let $\lambda$ be the monotone eigenvalue of $K_{c,\theta}(T)$ nearer to 0. Thus $-\lambda c$ is the Perron–Frobenius eigenvalue of $\frac{1}{2}\lambda^2A+\theta Q+R$; let $v_\lambda$ be the corresponding eigenvector with $v_\lambda(1)=1$. Define*

(1.40) $$Z_\lambda(t):=\sum_{k=1}^{N(t)} v_\lambda(Y_k(t))\exp\{\lambda[X_k(t)+ct]\}.$$

*Then $Z_\lambda$ is a true (not just a local) martingale, and $Z_\lambda(t)$ converges to a limit $Z_\lambda(\infty)$ almost surely and in $\mathcal{L}^1$. Moreover, $\mathbb{P}_{x,y}(Z_\lambda(\infty)>0)=1$ for all $x$ and $y$.*

Note that the next results give information on the PDE (1.2), not only on the travelling-wave equation (1.3). A more complete study of the PDE will be given elsewhere by one of us (J.W.).

(1.41) **Theorem.** *Continue with the assumptions and notation of Theorem 1.39. If $u$ satisfies the coupled reaction-diffusion system (1.2) and $u(\cdot,\cdot,\cdot)\in[0,1]$, and if also, for $y\in I$,*

(1.42) $$1-u(0,x,y)\sim v_\lambda(y)\mathrm{e}^{\lambda x}\qquad (x\to\infty),$$

*then, as $t\to\infty$,*

$$u(t,x+ct,y)\to w(x,y),$$

*where*

$$(1.43)\qquad w(x,y) := \mathbb{E}_{x,y}\exp[-Z_\lambda(\infty)].$$

*This function $w$ satisfies the travelling-wave equation (1.3) and is, modulo translations, the unique monotonic wave of speed $c$ from $S$ to $T$.*

(1.44) **Theorem.** *As $t \to \infty$, we have almost surely (a.s.)*

$$(1.45)\qquad t^{-1}L(t) \to -c(\theta), \text{ where } L(t) := \inf_{k \leqslant N(t)} X_k(t).$$

*If $u$ satisfies the coupled reaction-diffusion system (1.2), and if $0 \leqslant u(t,x) \leqslant 1$ for $t \geqslant 0$ and $x \in \mathbb{R}$ and if also*

$$(1.46)\qquad u(0,x,y) = \begin{cases} 1 & \text{if } x > 0, \\ \frac{1}{2} & \text{if } x = 0, \\ 0 & \text{if } x < 0, \end{cases}$$

*then for $t > 0$, $u(t,x,y) = \mathbb{P}_{x,y}[L(t) > 0]$, and $u$ is an approximate travelling wave of speed $c(\theta)$ in the sense that*

$$(1.47)\qquad u(t, x+\gamma t, y) \to \begin{cases} 1 & \text{if } \gamma > c(\theta), \\ 0 & \text{if } \gamma < c(\theta). \end{cases}$$

For the remainder of this subsection, we consider the case when $X_1(0) = 0$ and $Y_1(0) = 1$: that is, we work with the $\mathbb{P}_{0,1}$ law: $\mathbb{P} := \mathbb{P}_{0,1}$.

We now give some rough – and very dangerous – large-deviation heuristics for (1.45). Let $m$ denote a probability measure on the type-space $I$. For a single particle performing a Markov chain on $I$ with $Q$-matrix $\theta Q$, the likelihood that it will spend time $tm(y)$ in $y$ by time $t$ is

$$\exp\{-t\theta I(m,Q)\}$$

in a sense made precise by large-deviation theory. For our $(N,X,Y)$ process, we expect the number of particles at time $t$ which have their 'type-times' in the proportion described by $m$ to be roughly

$$\exp\{t[m(r) - \theta I(m,Q)]\};$$

and at time $t$, each of these particles (conditionally) has the Gaussian distribution of mean 0 and variance $m(a)t$. The total number of particles near $\gamma t$ is therefore roughly

$$\exp\{t[m(r) - \theta I(m,Q) - \gamma^2/2m(a)]\},$$

so that we expect to find particles near $\gamma t$ if and only if

$$\gamma^2 \leqslant \sup_m\{2m(a)\,[m(r) - \theta I(m,Q)]\}.$$

The problem with the above argument is not one of fussy details of rigour. For example, Lemma 4.7 below makes some of the reasoning precise. The problem is that, if for example, the initial situation is that there is one particle of type

1 situated at a randomly chosen point on $R$ with density $\frac{1}{2}\beta\exp(-|\beta x|)$, where $0 < \beta < |\lambda(\theta)|$, then we would get the *wrong* answer from the above heuristics:

(1.48) *'expectation waves' and 'particle wave-fronts' can move at different speeds*,

and it can be very difficult to decide when the answer given by the above heuristics is correct.

The following theorem explains the structure of our proof of Theorem 1.44.

(1.49) **Theorem.** (i) *Let $c$, $\lambda$ and $Z_\lambda$ be as in Theorem 1.39, and let $\Lambda_{\mathrm{PF}}(\lambda)$ be the Perron–Frobenius eigenvalue of $\frac{1}{2}\lambda^2 A + \theta Q + R$. The fact that $Z_\lambda$ is a martingale implies that*

(1.50) $$\liminf_{t\to\infty} t^{-1}L(t) \geqslant \lambda^{-1}\Lambda_{\mathrm{PF}}(\lambda) \quad \text{(a.s.)}.$$

(ii) *Since $Z_\lambda(\infty)$ exists in $\mathcal{L}^1$ and $Z_\lambda(0) = 1$, we can define a measure $Q_\lambda$ equivalent to $\mathbb{P}$ on $\mathcal{F}_\infty$ by*

(1.51) $$\mathrm{d}Q_\lambda/\mathrm{d}\mathbb{P} = Z_\lambda(\infty) \text{ on } \mathcal{F}_\infty, \text{ whence } \mathrm{d}Q_\lambda/\mathrm{d}\mathbb{P} = Z_\lambda(t) \text{ on } \mathcal{F}_t.$$

*Then*

(1.52) $$M_\lambda(t) := Z_\lambda(t)^{-1}\frac{\partial}{\partial\lambda}Z_\lambda(t)$$

*defines a $Q_\lambda$-martingale, and*

(1.53) $$t^{-1}M_\lambda(t) \to 0 \quad \text{(a.s.)}.$$

*This implies that*

(1.54) $$\limsup_{t\to\infty} t^{-1}L(t) \leqslant \frac{\partial}{\partial\lambda}\Lambda_{\mathrm{PF}}(\lambda) \quad \text{(a.s.)}.$$

(iii) *As $c \downarrow c(\theta)$, we have*

(1.55) $$\lambda^{-1}\Lambda_{\mathrm{PF}}(\lambda) \to -c(\theta) \quad \text{and} \quad \frac{\partial}{\partial\lambda}\Lambda_{\mathrm{PF}}(\lambda) \to -c(\theta),$$

*so that (1.45) follows.*

## 2. Proofs of algebraic results

### (a) *Proof of Lemma 1.11*

The idea is to extend results from $\theta = 0$ to other values of $\theta$.

*Part* (i). The characteristic polynomial of $K(S)$ is

$$\chi^S_{c,\theta}(\lambda) := \delta_1\delta_2[F^S_1(\lambda)F^S_2(\lambda) - \theta^2 q_1 q_2],$$

where

$$F^S_i(\lambda) = \tfrac{1}{2}a_i\lambda^2 + c\lambda - r_i - \theta q_i.$$

When $\theta = 0$, $\chi^S_{c,\theta}$ has two real positive roots and two real negative roots. For $\theta > 0$, $\chi^S_{c,\theta}(0) \neq 0$, and for $\tau > 0$, we have $\mathrm{Im}\,\chi^S_{c,\theta}(\mathrm{i}\tau) < 0$. We see that as $\theta$ increases from 0, no root of $\chi^S_{c,\theta}$ can cross the imaginary axis. Thus $d_{\mathrm{u}}(S) = 2$ for every $\theta > 0$ and every $c$. ∎

*Part* (ii). The characteristic polynomial of $K_{c,\theta}(T)$ is

$$\chi^T_{c,\theta}(\lambda) = \delta_1\delta_2 H(\lambda, c, \theta) = \delta_1\delta_2[F_1(\lambda, c, \theta)F_2(\lambda, c, \theta) - \theta^2 q_1 q_2], \tag{2.1}$$

where

$$F_i(\lambda, c, \theta) = \tfrac{1}{2}a_i\lambda^2 + c\lambda + r_i - \theta q_i. \tag{2.2}$$

When $\theta = 0$ and $c > 0$, all four roots of $\chi^T_{c,\theta}$ have negative real part. Suppose that for some $\theta > 0$, $\chi^T_{c,\theta}$ has an imaginary root $\lambda = \mathrm{i}\tau$, where $\tau \neq 0$. Then on looking at imaginary parts, we conclude that

$$(-\tfrac{1}{2}a_1\tau^2 + r_1 - \theta q_1) + (-\tfrac{1}{2}a_2\tau^2 + r_2 - \theta q_2) = 0,$$

so that the bracketed terms have opposite signs; and looking at real parts, we have

$$(-\tfrac{1}{2}a_1\tau^2 + r_1 - \theta q_1)(-\tfrac{1}{2}a_2\tau^2 + r_2 - \theta q_2) - c^2\tau^2 - \theta^2 q_1 q_2 = 0,$$

which is now clearly impossible. Hence $\chi^T_{c,\theta}$ cannot have a non-zero imaginary root.

We see that $\lambda = 0$ is a root of $\chi^T_{c,\theta}$ if and only if

$$\theta = \theta_0 := r_1 r_2/(q_1 r_2 + q_2 r_1) = 1/(\rho_1 + \rho_2),$$

and that the derivative of $\chi^T_{c,\theta}(\lambda)$ at $\lambda = 0$ is then strictly positive, so that 0 is a simple root. Hence, $d_s(T) = 4$ for $\theta < \theta_0$, and $d_s(T) = 3$ for $\theta \geqslant \theta_0$. ∎

*Part* (iii). The characteristic polynomial of $K(E_+)$ is

$$\delta_1\delta_2[\tilde{F}_1(\lambda)\tilde{F}_2(\lambda) - \theta^2 q_1 q_2],$$

where

$$\tilde{F}_1(\lambda, c, \theta) := \tfrac{1}{2}a_1\lambda^2 + c\lambda + 2r_1\sqrt{\Delta} + \theta q_1,$$
$$\tilde{F}_2(\lambda, c, \theta) := \tfrac{1}{2}a_2\lambda^2 + c\lambda - 2r_2\sqrt{\Delta} + \theta q_2.$$

The proof now proceeds along similar lines to that of part (ii), and is skipped.

### (*b*) *Proof of Lemma 1.13*

We continue to use the notation in (2.1) and (2.2). If $v$ is an eigenvector of $K_{c,\theta}(T)$ corresponding to $\lambda$, then

$$F_1(\lambda, c, \theta)v_1 + \theta q_1 v_2 = 0.$$

We learn two things from this: firstly, that the eigenspace corresponding to an eigenvalue has dimension 1; and secondly, that $\lambda$ is a stable monotone eigenvalue of $K_{c,\theta}(T)$ if and only if $\lambda$ is a real negative eigenvalue with $F_i(\lambda, c, \theta) < 0$ for one, then both, of $i = 1, 2$.

*Suppose that $K_{c,\theta}(T)$ has a stable monotone eigenvalue* $\lambda$. Then, since (for $i = 1, 2$) $F_i(\lambda, c, \theta) < 0$, the quadratic $F_i(\cdot, c, \theta)$ has real roots $\mu_i, \nu_i$ with $\mu_i < \nu_i$. Set $\mu_+ := \max(\mu_1, \mu_2)$ and $\mu_- := \min(\mu_1, \mu_2)$; and define $\nu_+$ and $\nu_-$ analogously. Then

$$F_1(\nu_+, c, \theta)F_2(\nu_+, c, \theta) = 0,$$

and, since both $F_i(\cdot, c, \theta)$ increase to the right of $\nu_+$, $K_{c,\theta}$ must have a unique (non-monotonic) eigenvalue to the right of $\nu_+$. By similar reasoning, $K_{c,\theta}(T)$




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must have a unique real non-monotone eigenvalue to the left of $\mu_-$. Since $K_{c,\theta}(T)$ has at least three real eigenvalues, it must have four real eigenvalues (if we count algebraic multiplicities); and the fourth real eigenvalue must lie within $(\mu_+, \nu_-)$ and must be stable monotone.

*Continue to assume that $K_{c,\theta}(T)$ has a stable monotone eigenvalue $\lambda$.* Now let $\tilde{c} > c$. Then, for $i = 1, 2$,

$$F_i(\lambda, \tilde{c}, \theta) < F_i(\lambda, c, \theta) < 0,$$

and

$$F_1(\lambda, \tilde{c}, \theta) F_2(\lambda, \tilde{c}, \theta) - \theta^2 q_1 q_2 > 0.$$

By continuity, there must exist $\tilde{\lambda} < \lambda$ with

$$F_i(\tilde{\lambda}, \tilde{c}, \theta) < 0, \qquad F_1(\tilde{\lambda}, \tilde{c}, \theta) F_2(\tilde{\lambda}, \tilde{c}, \theta) - \theta^2 q_1 q_2 = 0,$$

so that $\tilde{\lambda}$ is a stable monotone eigenvalue of $K_{\tilde{c},\theta}(T)$.

Note that the argument which we have just given establishes result (1.14). For any fixed $\lambda < 0$, we can choose a large $c$ such that at $(\lambda, c, \theta)$, we have $F_i < 0$ $(i = 1, 2)$ and $F_1 F_2 - \theta^2 q_1 q_2 > 0$, so that $c > c(\theta)$. The finiteness of

$$c(\theta) := \inf\{c > 0 : K_{c,\theta}(T) \text{ has a stable monotone eigenvalue}\}$$

is assured.

We now prove that $c(\theta) > 0$. Let

$$\ell_i(\theta) := \inf_{\lambda} F_i(\lambda, c, \theta) = -\tfrac{1}{2} c^2 / a_i + r_i - \theta q_i.$$

Then it follows from (1.14) that if $c > c(\theta)$, then

$$\ell_i < 0 \quad (i = 1, 2), \qquad \ell_1 \ell_2 > \theta^2 q_1 q_2,$$

so that

$$c^2 > 2a_i(r_i - \theta q_i) \quad (i = 1, 2)$$

and

$$[c^2 - 2a_1(r_1 - \theta q_1)][c^2 - 2a_2(r_2 - \theta q_2)] > 4\theta^2 a_1 a_2 q_1 q_2.$$

If it were the case that $c(\theta) = 0$, then we could let $c \downarrow 0$ in the inequalities just derived to obtain

$$r_i \leqslant \theta q_i \quad (i = 1, 2), \qquad (r_1 - \theta q_1)(r_2 - \theta q_2) \geqslant \theta^2 q_1 q_2,$$

yielding the contradiction

$$\rho_i \theta \geqslant 1 \quad (i = 1, 2), \qquad 1 \geqslant (\rho_1 + \rho_2)\theta.$$

(Recall that $\rho_i = q_i / r_i$.) Hence, $c(\theta) > 0$ for every $\theta$.

The rest is easy. ■

### (c) *Proof of Lemma 1.19*

Parts (i) and (ii) of Lemma 1.19 really are an immediate consequence of definitions and the uniqueness of the Perron–Frobenius eigenvalue of matrices with positive off-diagonal elements. As already stated, Part (iii) of Lemma 1.19 is a trivial case of a standard Donsker–Varadhan result on Legendre transformations

(Ellis 1985); and in this simple case the reader can prove it directly as an exercise.

*Proof of Lemma* 1.19(iv). Set

$$\phi(m_1) := m(r) - \theta I(m,Q) = m(r-\theta q) + 2\theta[q_1 q_2 m_1(1-m_1)]^{1/2},$$
$$\Phi(\mu, m_1) := \tfrac{1}{2}\mu m(a) + \mu^{-1}\phi(m_1) = \tfrac{1}{2}\mu a_2 + \tfrac{1}{2}\mu m_1(a_1-a_2) + \mu^{-1}\phi(m_1),$$
$$c(\theta) := \inf_{\mu>0} \sup_{m_1\in[0,1]} \Phi(\mu, m_1), \qquad d(\theta) := \sup_{m_1\in[0,1]} \inf_{\mu>0} \Phi(\mu, m_1).$$

Of course, the fact that $c(\theta) \geqslant d(\theta)$ follows from the definitions.

Let $\mathcal{M} := \{m_1 \in [0,1] : \phi(m_1) \geqslant 0\}$. It is clear that for $m_1 \notin \mathcal{M}$, we have $\inf\{\Phi(\mu, m_1) : \mu > 0\} = -\infty$, whence

$$d(\theta) = \sup_{m_1\in\mathcal{M}} \inf_{\mu>0} \Phi(\mu, m_1).$$

Because $\Phi(\cdot, m_1)$ is convex on $(0,\infty)$ for $m_1 \in \mathcal{M}$, and $\Phi(\mu,\cdot)$ is concave on $\mathcal{M}$ for $\mu \in (0,\infty)$, standard theory (see Ekeland & Temam 1976, ch. VI, § 2.2) implies that

$$d(\theta) = \inf_{\mu>0} \sup_{m_1\in\mathcal{M}} \Phi(\mu, m_1).$$

Now, $\phi$ is strictly concave on $[0,1]$ with infinite derivatives at 0 and 1. Hence, for fixed $\mu > 0$, there exists a unique $\hat{m}_1(\mu)$ in $[0,1]$ such that

$$\Phi(\mu, \hat{m}_1(\mu)) = \sup_{m_1\in[0,1]} \Phi(\mu, m_1).$$

We have

$$\tfrac{1}{2}\mu^2(a_1-a_2) + \phi'(\hat{m}_1(\mu)) = 0,$$

where $'$ denotes differentiation with respect to $m_1$; and since $\phi'' < 0$ on $(0,1)$ the function $\hat{m}_1(\mu)$ is real analytic. Moreover,

$$\mu(a_1-a_2) + \phi''(\hat{m}_1(\mu))\frac{\mathrm{d}\hat{m}_1}{\mathrm{d}\mu}(\mu) = 0,$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}\mu}[\phi(\hat{m}_1(\mu))] = \phi'(\hat{m}_1(\mu))\hat{m}_1'(\mu) = \tfrac{1}{2}\mu^3(a_1-a_2)^2/\phi''(\hat{m}(\mu)) < 0.$$

Now, since $I(\pi,Q) = 0$, $\phi(\pi_1) = \pi(r) > 0$, and it is clear that if we define $\mathcal{L} := \{\mu : \phi(\hat{m}_1(\mu)) \leqslant 0\}$, then either $\mathcal{L} = [\mu_0,\infty)$ for some $\mu_0 > 0$ or $\mathcal{L} = \emptyset$. Now if $\mathcal{L} = [\mu_0,\infty)$ and $\mu > \mu_0$, then

$$\frac{\mathrm{d}}{\mathrm{d}\mu}[\Phi(\mu, \hat{m}_1(\mu))] = \tfrac{1}{2}\hat{m}(\mu)(a) - \phi(\hat{m}_1(\mu))\mu^{-2} > 0,$$

so that

$$c(\theta) = \inf_{\mu\leqslant\mu_0} \Phi(\mu, \hat{m}_1(\mu)).$$

Now, since $\phi$ and $\Phi$ are concave functions of $m$, for $\mu \geqslant \mu_0$, $\sup_{m_1\in\mathcal{M}} \Phi(\mu, m_1)$ is attained at a point $m_1 = \tilde{m}_1$, where $\phi(\tilde{m}_1) = 0$ and $\tilde{m}_1$ is independent of $\mu$.

Hence,

$$\inf_{\mu\geqslant\mu_0}\sup_{m_1\in\mathcal{M}}\Phi(\mu,m_1)=\inf_{\mu\geqslant\mu_0}\tfrac{1}{2}\mu\tilde{m}(a)=\tfrac{1}{2}\mu_0\tilde{m}(a)=\sup_{m_1\in\mathcal{M}}\Phi(\mu_0,m_1).$$

Therefore,

$$\begin{aligned}d(\theta)&=\inf_{\mu>0}\sup_{m_1\in\mathcal{M}}\Phi(\mu,m_1)=\inf_{\mu\leqslant\mu_0}\sup_{m_1\in\mathcal{M}}\Phi(\mu,m_1)\\&=\inf_{\mu\leqslant\mu_0}\sup_{m_1\in[0,1]}\Phi(\mu,m_1)=c(\theta).\end{aligned}$$

In the remaining case when $\mathcal{L}=\emptyset$, the result is obvious. ■

### (d) *Proof of Lemma 1.24*

The derivative with respect to $(\lambda,c)$ of the map

$$(\lambda,c,\theta)\mapsto(H(\lambda,c,\theta),H_\lambda(\lambda,c,\theta))$$

has Jacobian

$$J=H_\lambda H_{\lambda c}-H_c H_{\lambda\lambda},\ \text{where } H_{\lambda c}:=\frac{\partial^2 H}{\partial\lambda\partial c},\ \text{etc.}$$

At $(\lambda(\theta),c(\theta),\theta)$, we have

$$\begin{aligned}H_c&=\lambda(F_1+F_2)>0,\ \text{since } \lambda,F_1,F_2<0,\\0=H_\lambda&=(a_1\lambda+c)F_2+(a_2\lambda+c)F_1,\ \text{so that } (a_1\lambda+c)(a_2\lambda+c)\leqslant 0,\\H_{\lambda\lambda}&=a_1F_2+a_2F_1+2(a_1\lambda+c)(a_2\lambda+c)<0.\end{aligned}$$

Hence, $J<0$ at $(\lambda(\theta),c(\theta),\theta)$. The smoothness of $c(\theta)$ and $\lambda(\theta)$ now follows from the Implicit Function Theorem.

The monotonicity of $c(\theta)$ is of course clear from (1.23). (An implicit formula for $c'(\theta)$, from which monotonicity is also immediate, may be obtained by differentiating the formulae in the above proof of smoothness.) The facts that $c(0+)=c_{\mathrm{F}}$ and $c(\infty)=c_{\mathrm{M}}$ are left to the reader.

### (e) *Further algebraic results*

Several other algebraic results play essential roles in the later analysis and probability. It seems good sense to collect them here, while $F_i(\lambda,c,\theta)$, $H(\lambda,c,\theta)$, etc., are fresh in the reader's mind.

The next lemma, though trivial in the light of what we now know, is worth stating separately.

(2.3) **Lemma.** *Suppose that $c>c(\theta)$. Let $\lambda$ and $\beta$ be the two stable monotone eigenvalues of $K_{c,\theta}(T)$, labelled so that $\beta<\lambda<0$. Thus $\lambda$ is the monotone eigenvalue of $K_{c,\theta}(T)$ closer to 0, and $\beta$ the eigenvalue further from 0. Then*

$$H(\mu,c,\theta)=(F_1F_2)(\mu,c,\theta)-\theta^2q_1q_2>0\quad\textit{for}\quad\beta<\mu<\lambda.$$

*Recall that $F_i(\lambda,c,\theta)<0$ for $i=1,2$.*

*Proof.* This is left to the reader. ■

We need the following result in our proof of $\mathcal{L}^1$ convergence of $Z_\lambda$ in Theorem 1.39.





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(2.4) **Lemma.** (i) *Suppose that* $c > c(\theta)$. *Let* $\lambda$ *be the stable monotone eigenvalue of* $K_{c,\theta}(T)$ *nearer to* 0. (*We know that* $-\lambda c$ *is the Perron–Frobenius eigenvalue of* $(\frac{1}{2}\lambda^2 A + \theta Q + R)$.) *For* $\mu < \lambda$ *with* $\mu$ *sufficiently close to* $\lambda$,

$$\Lambda_{\mathrm{PF}}(\tfrac{1}{2}\mu^2 A + \theta Q + R) = -\mu c_1(\mu) \text{ for some } c_1(\mu) < c.$$

(ii) *Part* (iii) *of Theorem 1.49 is true.*

*Proof of* (i). Locally for $\mu$ near $\lambda$, $c_1(\mu)$ is the unique solution of $H(\mu, c_1(\mu), \theta) = 0$ by the Implicit Function Theorem, because $\partial H(\lambda, c, \theta)/\partial\mu < 0$ by Lemma 2.3. Thus,

$$\frac{\partial H}{\partial \mu} + \frac{\partial H}{\partial c} c_1'(\mu) = 0.$$

At $(\lambda, c, \theta)$, we have $\partial H/\partial\mu < 0$, and

$$\frac{\partial H}{\partial c} = \lambda(F_1 + F_2) > 0,$$

since $\lambda < 0$, $F_1 < 0$ and $F_2 < 0$. Hence $c_1'(\lambda) > 0$, and part (i) follows. Part (ii) is left to the reader. ■

For the probabilistic proof of uniqueness modulo translation of monotonic travelling waves from $S$ to $T$, the next lemma is important.

(2.5) **Lemma.** *Suppose that* $c > c(\theta)$. *Let* $\beta$ *be the stable monotone eigenvalue of* $K_{c,\theta}(T)$ *further from* 0. *Then, for* $\alpha > \beta$ *with* $\alpha$ *sufficiently close to* $\beta$, *the only non-negative* $I$*-vector* $g$ *such that*

$$(\tfrac{1}{2}\alpha^2 A + \alpha c I + \theta Q + R) g \geqslant 0 \tag{2.6}$$

*is the zero vector:* $g = 0$.

*Proof.* We have

$$\tfrac{1}{2}\alpha^2 A + \alpha c I + \theta Q + R = \begin{pmatrix} F_1 & \theta q_1 \\ \theta q_2 & F_2 \end{pmatrix} (\alpha, c, \theta)$$

with determinant $H(\alpha, c, \theta)$. By Lemma 2.3 and the fact that $H(\beta, c, \theta) = 0$, we can choose $\alpha > \beta$ so close to $\beta$ that

$$F_i(\alpha, c, \theta) < 0 \quad (i = 1, 2), \qquad H(\alpha, c, \theta) > 0.$$

Then the inverse

$$G = H(\alpha, c, \theta)^{-1} \begin{pmatrix} F_2 & -\theta q_1 \\ -\theta q_2 & F_1 \end{pmatrix} (\alpha, c, \theta)$$

of $\frac{1}{2}\alpha^2 A + \alpha c I + \theta Q + R$ has all entries negative. Hence, from (2.6) and the non-negativity of $g$, we see that $g$ must equal 0. ■

Finally, we need the following result in showing that

$$\mathbb{P}_{x,y}[Z_\lambda(\infty) = 0] = 0 \text{ or } 1.$$

(2.7) **Lemma.** *If* $w$ *is a vector on* $I$ *such that* $0 \leqslant w \leqslant 1$ *and*

$$R(w^2) = (R - \theta Q) w,$$

*then either $w = 1$ on $I$ or $w = 0$ on $I$.*

*Proof.* This lemma is easily proved assuming only that $I$ is finite, $Q$ is an irreducible $Q$-matrix on $I$, $\theta > 0$, and $R$ is a positive diagonal matrix. For we have

$$w^2 = (I - H)w, \qquad w = \Gamma(w^2)$$

where $H$ is the $Q$-matrix $\theta R^{-1}Q$ and $\Gamma$ is the irreducible stochastic matrix $(I - H)^{-1}$. Thus,

$$\sup w \leqslant \sup(w^2) = (\sup w)^2,$$

so that $\sup w = 0$ or 1. If $w(i_0) = 1$, then $w(j)$ must be 1 for every $j$ such that $\Gamma^n(i_0, j) > 0$ for some $n$. ∎

### (*f*) *Proof of Lemma 1.29*

The first equality in Lemma 1.29 is a consequence of the equality of two from $E_+, E_-$ and $T$ when $\theta = \theta_0$ or $\theta^*$. The inequality for $c'(\theta)$ is known from Lemma (1.24).

If $(\alpha_1^\pm, \alpha_2^\pm, 0, 0)$ denote the coordinates of $E_\pm$ given by (1.6) then the characteristic equation for the eigenvalues of the linearization at $E_\pm$ may be written

$$F_1^\pm(\lambda, c, \theta)F_2^\pm(\lambda, c, \theta) = \theta^2 q_1 q_2,$$

where

$$F_i^\pm(\lambda, c, \theta) = \tfrac{1}{2}\lambda^2 a_i + \lambda c + r_i(2\alpha_i^\pm - 1) - \theta q_i.$$

Let $\theta > 0$. Then $c_\pm(\theta)$ is characterized as the unique positive $c$ for which this equation has a double negative root with $F_i^\pm < 0$. Recall that $c(\theta)$ is characterized like this too when $(\alpha_1, \alpha_2) = (1, 1)$ and the proof, by the Implicit Function Theorem, that $c(\theta)$ is a real-analytic function of $\theta$ may be applied without change to yield that $c_\pm(\theta)$ is smooth as well. If $\lambda^\pm(\theta)$ denotes the negative double characteristic eigenvalue when $c = c_\pm(\theta)$ then $\lambda^\pm$ also depends smoothly on $\theta$. Differentiating the characteristic polynomial with respect to $\theta$, using the fact that it has a double root at $\lambda = \lambda^\pm(\theta)$, gives

$$\begin{aligned} 0 &= \lambda^\pm(\theta)(F_1^\pm + F_2^\pm)\frac{\mathrm{d}c_\pm}{\mathrm{d}\theta} + F_1^\pm \frac{\mathrm{d}}{\mathrm{d}\theta}[r_2(2\alpha_2^\pm(\theta) - 1) - \theta q_2] \\ &\quad + F_2^\pm \frac{\mathrm{d}}{\mathrm{d}\theta}[r_1(2\alpha_1^\pm(\theta) - 1) - \theta q_1] - 2\theta q_1 q_2 \\ &= \lambda^\pm(\theta)(F_1^\pm + F_2^\pm)\frac{\mathrm{d}c_\pm}{\mathrm{d}\theta} + q_2 F_1^\pm \left(1 \pm \frac{4\theta\rho_1}{\sqrt{1 - 4\theta^2\rho_1\rho_2}}\right) \\ &\quad + q_1 F_2^\pm \left(1 \mp \frac{4\theta\rho_2}{\sqrt{1 - 4\theta^2\rho_1\rho_2}}\right) - 2\theta q_1 q_2. \end{aligned}$$

Since $F_i^\pm(\lambda^\pm(\theta), c_\pm(\theta), \theta) < 0$ and $\lambda^\pm(\theta) < 0$, it follows that

$$\frac{\mathrm{d}c_\pm}{\mathrm{d}\theta} > 0 \ \text{ if } \ 2\theta < \sqrt{\frac{1}{4\rho_1^2 + \rho_1\rho_2}}.$$

*Now we consider the case $\theta = 0$.* Let

$$Q_i = \tfrac{1}{2}\lambda^2 a_i + \lambda c + r_i, \quad i = 1, 2,$$

$$k_1 = \frac{a_2}{a_1} + \frac{r_2}{r_1}, \quad k_2 = \frac{a_1}{a_2} + \frac{r_1}{r_2}.$$

That we have both $k_1 < 2$ and $k_2 < 2$ is impossible; and if both $k_1 > 2$ and $k_2 > 2$, then 1 is strictly between $a_2/a_1$ and $r_2/r_1$.

*Suppose first that both $k_1$ and $k_2$ exceed* 2; and without loss of generality that $(a_1/a_2) > 1 > (r_1/r_2)$. Let

$$\tilde{\lambda} = -\sqrt{\frac{2(r_2 - r_1)}{a_1 - a_2}} < 0 \quad \text{and} \quad \tilde{c} = \frac{a_1 r_2 - a_2 r_1}{\sqrt{2(r_2 - r_1)(a_1 - a_2)}} > 0.$$

Then, when $(\lambda, c) = (\tilde{\lambda}, \tilde{c})$,

$$0 = Q_1 = Q_2 = Q_1 Q_2 = \frac{\partial}{\partial \lambda}(Q_1 Q_2),$$

and

$$\frac{\partial^2}{\partial \lambda^2}(Q_1 Q_2) = 2\frac{\partial Q_1}{\partial \lambda}\frac{\partial Q_2}{\partial \lambda} = \frac{-a_1 r_1 a_2 r_2 (2 - k_1)(2 - k_2)}{(r_2 - r_1)(a_1 - a_2)} < 0.$$

Hence $c(0) = \tilde{c}$ in this case. Also, when $(\lambda, c) = (\tilde{\lambda}, \tilde{c})$,

$$(Q_1 - 2r_1)Q_2 = 0, \qquad \frac{\partial}{\partial \lambda}((Q_1 - 2r_1)Q_2) = (Q_1 - 2r_1)\frac{\partial Q_2}{\partial \lambda} < 0,$$

and

$$(Q_1 - 2r_1) < 0, Q_2 = 0, \qquad \frac{\partial Q_2}{\partial \lambda} > 0.$$

Hence $c_-(0) < c(0)$ and similarly $c_+(0) < c(0)$ in this case.

*Now suppose that $k_1 < 2 \leqslant k_2$.* Let

$$\hat{\lambda} = -\sqrt{\frac{2r_1}{a_1}}, \quad \hat{c} = \sqrt{2a_1 r_1}.$$

Then, when $(\lambda, c) = (\hat{\lambda}, \hat{c})$,

$$Q_1 = \frac{\partial Q_1}{\partial \lambda} = 0 \quad \text{and} \quad \frac{\partial^2}{\partial \lambda^2}(Q_1 Q_2) = r_1 a_1 \left(\frac{a_2}{a_1} + \frac{r_2}{r_1} - 2\right) \leqslant 0.$$

Hence $c(0) = \sqrt{2a_1 r_1}$. Also when $(\lambda, c) = (\hat{\lambda}, \hat{c})$,

$$Q_1(Q_2 - 2r_2) = \frac{\partial}{\partial \lambda}(Q_1(Q_2 - 2r_2)) = 0 \quad \text{and} \quad \frac{\partial^2}{\partial \lambda^2}(Q_1(Q_2 - 2r_2)) < 0.$$

Hence $c_+(0) = \sqrt{2a_1 r_1} = c(\theta)$. Now note that

$$(Q_1 - 2r_1) < 0, \quad Q_2 < 0, \quad \hat{\lambda} < 0$$

and hence $c_-(0) < c_+(0) = c(0)$ in this case.

*The case when $k_2 < 2 \leqslant k_1$* is similar and yields $c_+(0) < c_-(0) = c(0)$.

*The case when $k_1 = k_2 = 2$* is immediate. ■





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## 3. Analytic proofs of existence and uniqueness results

The proof of Theorem 1.30 uses a shooting argument and proceeds via a sequence of lemmas, some of which are of independent interest. For example, in passing we give an elementary proof of the rate at which travelling waves converge to $T$ or $E_{\pm}$. We often have in mind the two-dimensional $(w_1, w_2)$ picture, so that an equilibrium will be written $(\alpha_1, \alpha_2)$ rather than $(\alpha_1, \alpha_2, 0, 0)$.

### (*a*) *Two key lemmas*

The following lemma implies that if $E$ in $\{E_+, E_-, T\}$ has a monotone eigenvalue, then no bounded monotone wave can pass through $E$ at a finite time. The identity used to prove it goes some way to explaining the dramatic character of the phase-space projections on $w$-space displayed in figures 3 and 4.

(3.1) **Lemma.** *Let $c > 0$ and $\theta > 0$ be such that for some $E \in \{E_+, E_-, T\}$, $K(E)$ has a stable monotone eigenvalue. Let $w$ be a solution of (1.3) on the interval $(-\infty, \hat{x})$ with $w(\hat{x}) = E$ and $w'(\hat{x}) \geqslant 0$. Then $(w(x), w'(x))$ is unbounded as $x \to -\infty$.*

*Proof.* Let the coordinates of $E$ be $(\alpha_1, \alpha_2)$ and write the solution in the statement of the lemma as $w = (\alpha_1, \alpha_2) + (u_1, u_2)$ where $(u_1, u_2) = u$. Then

$$\tfrac{1}{2}Au'' + cu' + R(Du + u^2) + \theta Qu = 0,$$

where $D := \operatorname{diag}(2\alpha_1 - 1, 2\alpha_2 - 1)$. Let $v > 0$ be the first two components of a monotone eigenvector corresponding to the stable monotone eigenvalue $\lambda < 0$ of $K(E)$. (Recall Definition 1.12.) Then

$$(\tfrac{1}{2}\lambda^2 A + \lambda c + RD + \theta Q)v = 0,$$

and it is easy to see that $\tilde{v} = (q_2 v_1, q_1 v_2)$ is a positive eigenvector of the transpose problem

$$(\tfrac{1}{2}\lambda^2 A + \lambda c + RD + \theta Q^{\mathrm{T}})\tilde{v} = 0$$

for the same eigenvalue $\lambda$. In particular, if

$$\phi(x) = \mathrm{e}^{-\lambda x}\tilde{v},$$

then

$$\tfrac{1}{2}A\phi'' - c\phi' + (RD + \theta Q^{\mathrm{T}})\phi = 0 \text{ on } \mathbb{R},$$

and, since $\lambda < 0$,

$$\phi(x) \to 0 \text{ as } x \to -\infty.$$

Now recall that, by definition, $u(\hat{x}) = 0$. Suppose (for the purpose of obtaining a contradiction) that $\{(u(x), u'(x)) : x \in (-\infty, \hat{x})\}$ is bounded. Then multiplying the equation for $u$ by $\phi$ and integrating on $(-\infty, \hat{x})$ gives

$$\tfrac{1}{2}\mathrm{e}^{-\lambda\hat{x}}\langle Au'(\hat{x}), \tilde{v}\rangle + \int_{-\infty}^{\hat{x}} \langle R(u^2(s)), \phi(s)\rangle \,\mathrm{d}s = 0.$$

Since, by hypothesis, the first term on the left-hand side is non-negative and the second is positive, this is a contradiction which proves the result. ∎

To apply certain standard theorems on differential equations, we need the following result which establishes convergence at exponential rate. It is worth noting that it holds for all $\theta > 0$, including the case $\theta = \theta_0$ when $(1,1,0,0)$ is not a hyperbolic equilibrium of (1.3).

(3.2) **Lemma.** *Suppose that $c \geqslant c(\theta)$ and $w$ is a solution of* (1.3) *such that*

$$w'(x) > 0 \text{ on} [x_0, \infty) \text{ and } w(x) \to T \text{ as } x \to \infty.$$

*If $w(x) = (1,1) + u(x)$, then $\|u(x)\| + \|u'(x)\| \leqslant k\mathrm{e}^{-\gamma x}, x \in [x_0, \infty)$ for some $\gamma, k > 0$. An analogous result when $c \geqslant c_\pm(\theta)$ and $w(x) \to E_\pm$ as $x \to \infty$ also holds for different positive constants $\gamma$ and $k$.*

*Proof.* The equation satisfied by $u$, where $w = (1,1) + u$, is

$$\tfrac{1}{2}Au'' + cu' + (R + \theta Q)u + Ru^2 = 0$$

where $u(x) < 0 < u'(x)$, $x \in [x_0, \infty)$, and $u(x) \to 0$ as $x \to \infty$. Let $\mu$ be the Perron–Frobenius eigenvalue of the operator $(R + \theta Q^{\mathrm{T}})$ with positive eigenvector $f$. In general, $\mu$ is merely the larger of the eigenvalues of $(R + \theta Q^{\mathrm{T}})$, but in this particular case an inner product of the equation $(R + \theta Q^{\mathrm{T}})f = \mu f$ with the positive vector $(1,1)$ yields that $\mu > 0$. Now

$$\tfrac{1}{2}\langle Au'', f\rangle + c\langle u', f\rangle + \mu\langle u, f\rangle + \langle Ru^2, f\rangle = 0, \qquad x \in [x_0, \infty).$$

Let $\epsilon \in (0, \mu)$. Then $\tilde{x} \in [x_0, \infty)$ can be chosen sufficiently large that $0 < \langle Ru^2, f\rangle < -\epsilon\langle u, f\rangle$, and hence

$$\tfrac{1}{2}\langle Au'', f\rangle + c\langle u', f\rangle + (\mu - \epsilon)\langle u, f\rangle > 0, \qquad x \in [\tilde{x}, \infty).$$

Let $\psi(x) = \langle Au(x), f\rangle$. Then since $c > 0$, $(\mu - \epsilon) > 0$, $u' > 0$ and $u < 0$, there exist $k > 0$, $h > 0$ such that

$$\psi'' + h\psi' + k\psi > 0, \qquad \psi < 0 < \psi', \qquad x \in [\tilde{x}, \infty)$$

and

$$\psi(x) \to 0 \quad \text{as} \quad x \to \infty.$$

The first of these inequalities may be rewritten

$$(\mathrm{e}^{hx}\psi')' + k\mathrm{e}^{hx}\psi > 0.$$

Now if $\phi$ is a solution of

$$(\mathrm{e}^{hx}\phi')' + k\mathrm{e}^{hx}\phi = 0,$$

then it is immediate, by Sturm's Comparison Theorem (Coddington & Levinson 1955, ch. 8, Theorem 1.1) and by the fact that $\psi < 0$ on $[x_0, \infty)$, that $\phi$ has at most one zero in $[\tilde{x}, \infty)$. In particular, the characteristic equation

$$\mu^2 + h\mu + k = 0$$

has real solutions and, since $h$ and $k$ are positive, both solutions are negative. Let them be denoted by $-\beta \leqslant -\gamma < 0$. Then

$$(\partial_x + \beta)(\partial_x + \gamma)\psi > 0, \qquad \psi < 0, \qquad x \in [\tilde{x}, \infty).$$

If, at any point $\hat{x} \in [\tilde{x}, \infty)$, $(\partial_x + \gamma)\psi(\hat{x}) \geqslant 0$, then it is immediate (using the integrating factor $\mathrm{e}^{\beta x}$) that $(\partial_x + \gamma)\psi(x) \geqslant 0$ for all $x \geqslant \hat{x}$. The integrating factor $\mathrm{e}^{\gamma x}$ now yields

$$\mathrm{e}^{\gamma x}\psi(x) \geqslant \mathrm{e}^{\gamma\hat{x}}\psi(\hat{x}) \quad \text{for all} \quad x \geqslant \hat{x},$$

whence, since $\psi < 0$,

$$|\langle Au(x), f\rangle| = |\psi(x)| \leqslant e^{\gamma(\hat{x}-x)}|\psi(\hat{x})|, \qquad x \geqslant \hat{x}.$$

The only other possibility is that $(\partial_x + \gamma)\psi < 0$ for all $x \in [\tilde{x}, \infty)$. In this case, since $e^{\beta x}(\partial_x + \gamma)\psi$ is increasing, we find that for some $k > 0$,

$$\psi' + \gamma\psi \geqslant -ke^{-\beta x}, \qquad x \geqslant \tilde{x},$$

whence

$$(e^{\gamma x}\psi)' \geqslant -ke^{(\gamma-\beta)x}.$$

Hence, since $\beta \geqslant \gamma$,

$$e^{\gamma x}\psi(x) \geqslant e^{\gamma\tilde{x}}\psi(\tilde{x}) - k\int_{\tilde{x}}^{x} e^{(\gamma-\beta)s}\, ds,$$

whence $\psi < 0$ and

$$|\psi(x)| \leqslant \begin{cases} \hat{k}e^{-\gamma x} & \text{if } \beta > \gamma, \\ \hat{k}xe^{-\gamma x} & \text{if } \beta = \gamma, \end{cases}$$

for some $\hat{k} > 0$. This shows that there exists $\gamma > 0$ such that

$$|u_1(x)| + |u_2(x)| \leqslant \text{const.} \times e^{-\gamma x} \quad \text{for} \quad x > 0.$$

Now one integration of (1.3) on $(x, \infty)$ yields the same result for $u_1'$ and $u_2'$. This completes the proof of the Lemma. ■

## (*b*) *Exploiting the maximum principle*

In the two-dimensional $w = (w_1, w_2)$-plane consider the parabolae,

$$P_1 : \theta q_1 w_2 - (r_1 + \theta q_1)w_1 + r_1 w_1^2 = 0,$$

$$P_2 : \theta q_2 w_1 - (r_2 + \theta q_2)w_2 + r_2 w_2^2 = 0,$$

and let $\Omega_i$ denote the open region in the first quadrant between $P_i$ and the $w_i$-axis, $i = 1, 2$. The point $T = (1, 1) \in P_1 \cap P_2$ for all $c$, $\theta > 0$ and $E_\pm \in P_1 \cap P_2$ if $0 < \theta \leqslant \theta^*$.

Note that the relative positions of $E_+$, $E_-$ and $T$ depend on the value of $\theta$ and that no two of them are commensurate with respect to the partial ordering on $\mathbb{R}^2$ induced by the positive quadrant. Let us denote by $E_1$ the element of $\{E_+, E_-, T\}$ with the largest $w_1$-component and by $E_3$ that with the largest $w_2$-component.

*The region* $\Sigma$. Now let $\tilde{\Sigma}$ denote the rectangle in the $w$-plane with two sides on the axes intersecting at 0, a side through $E_1$ parallel to $\{w_1 = 0\}$ and one through $E_3$ parallel to $\{w_2 = 0\}$. Let $\Sigma$ denote the open convex subset of $\tilde{\Sigma}$ whose boundary comprises four straight line segments from $\partial\tilde{\Sigma}$, a parabolic segment from $P_1$ joining $E_1$ to $E_2$ and a parabolic segment from $P_2$ joining $E_2$ to $E_3$. Thus the boundary of $\Sigma$ always has four straight line segments: in addition it has two parabolic components when $E_+$, $E_-$ and $T$ are distinct, one parabolic component when two of $E_+$, $E_-$ and $T$ coincide and no parabolic component when $\theta > \theta^*$. (See figure 6.) Also $\Sigma$ consists of the union of three sets: $\omega_2 = \Sigma \cap \Omega_1 \cap \Omega_2$, a relatively closed component $\omega_3$ whose boundary intersects $\{w_1 = 0\}$ away from





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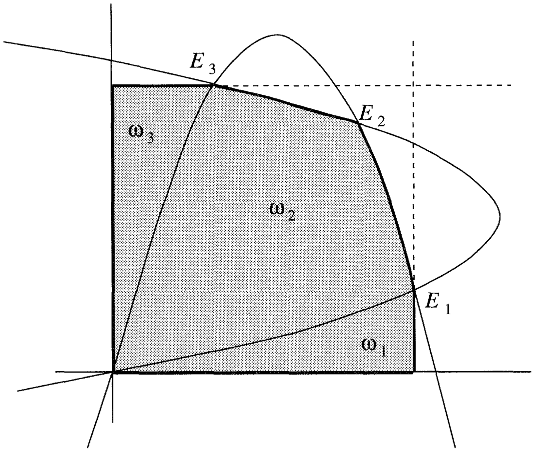


Figure 6.

the origin and a relatively closed component $\omega_1$ whose boundary intersects $\{w_2 = 0\}$ away from the origin. Note that

$$(r_1 + \theta q_1)w_1 - r_1 w_1^2 - \theta q_1 w_2 > 0, \quad (w_1, w_2) \in \omega_2 \cup \omega_1,$$

$$(r_2 + \theta q_2)w_2 - r_2 w_2^2 - \theta q_2 w_1 > 0, \quad (w_1, w_2) \in \omega_2 \cup \omega_3;$$

and hence, if $w = (w_1, w_2)$ satisfies (1.3) then, by the maximum principle,

$w_1$ has no local maximum at $x$ if $w(x) \in \omega_2 \cup \omega_1$,
$w_2$ has no local maximum at $x$ if $w(x) \in \omega_2 \cup \omega_3$,
$w_1$ has no local minimum at $x$ if $w(x) \in \omega_3 \setminus \overline{\omega}_2$,
$w_2$ has no local minimum if $w(x) \in \omega_1 \setminus \overline{\omega}_2$.

These observations lead to the following lemma.

(3.3) **Lemma.** *Suppose $w$ is a solution of (1.3) with $w(x) \in \Sigma$ and $w'(x) > 0$ for all $x \in (-\infty, \hat{x})$. If $w(x) \in \Sigma$ for all $x \in \mathbb{R}$ then $w'(x) > 0$ for all $x \in \mathbb{R}$ and $w(x)$ converges to $T, E_+$ or $E_-$ as $x \to \infty$. If $w(\tilde{x}) \in \partial\Sigma$ for some $\tilde{x} > \hat{x}$, and $\tilde{x}$ is the smallest such $x$, then at most one of $w_1(x)$ and $w_2(x)$ has exactly one local maximum, and the other has no turning point, in $(\hat{x}, \tilde{x}]$. Moreover $w(x) \notin \overline{\Sigma}$ for $x \in (\tilde{x}, \tilde{x} + \epsilon)$ for some $\epsilon > 0$. If $w$ is not monotone on $[\hat{x}, \tilde{x})$ then $w(\tilde{x})$ is a point of $\partial\Sigma$ which does not lie in the closed portion of the boundary which joins $E_1$ and $E_3$ and passes through $E_2$.*

*Proof.* Suppose that $w(\tilde{x}) \in \partial\Sigma, \tilde{x} > \hat{x}$ and $\tilde{x}$ is the smallest such $x$. Let $x^*$ be the smallest $x \in [\hat{x}, \tilde{x})$ at which one of $w_1(x)$ or $w_2(x)$ has a maximum, if such exist. Say, without loss of generality, that $w_1$ has a local maximum at $x^*$. Then $w(x^*) \in \omega_3$ and $w_2'(x^*) > 0$ by the maximum principle and the remarks preceding the statement of the Lemma. Consequently, $w_1'(x) < 0$ and $w_2'(x) > 0$ for $x \in (x^*, x^* + \epsilon)$ for some $\epsilon > 0$. However, $w_2$ has no local maximum and $w_1$ has no local minimum in the interior of $\omega_3$. Therefore $w_1$ and $w_2$ have no further turning points until they encounter the boundary of $\Sigma$ with non-zero derivative directed outwards at a point of $\partial\omega_3$. (See figure 6, and also figures 3 and 4.) Clearly, for some $\epsilon > 0$, $w(x) \notin \overline{\Sigma}$ for all $x \in (\tilde{x}, \tilde{x} + \epsilon)$ in this case.

An identical argument when $w_2$ has a turning point before $w_1$ yields an analogous result. If neither $w_1$ nor $w_2$ has a turning point in $[\hat{x}, \tilde{x})$ then it follows at once from the use of a integrating factor in (1.3) that at least one of $w_1'$ or $w_2'$ is non-zero at $\tilde{x}$ and that $w(x) \notin \overline{\Sigma}$ for all $x \in (\tilde{x}, \tilde{x}+\epsilon)$ for some $\epsilon > 0$ because the outward component of $w'(\tilde{x})$ to $\partial\Sigma$ is non-zero.

Now suppose that $w(x) \in \Sigma$ for all $x \in \mathbb{R}$. It is immediate from the preceding argument that $w' > 0$ on $\mathbb{R}$. As $w(x)$ is bounded and monotone it follows that $w(x)$ converges, to $w^* \in \mathbb{R}^2$ say, as $x \to \infty$. Since $w$ satisfies (1.3) we conclude that $w^* \in \{E_+, E_-, T\}$. ∎

### (c) *The unstable manifold at S*

To establish the main result of § 1 *c* we shall use the preceding lemma in conjunction with a shooting argument in which initial data is chosen on the unstable manifold of the zero equilibrium $S$. This ensures automatically that as $x \to -\infty$ the solution $w$ of (1.3) converges to $S$; and it then suffices to show that the initial data can be chosen so that the trajectory tends monotonically to one of the other equilibria as $x \to +\infty$. Note that a monotone travelling wave from $S$ to $T$ ($E_+$ or $E_-$) is possible only if $T$ ($E_+$ or $E_-$) has stable monotonic eigenvalues. Note also that, since the characteristic polynomial of $K(S)$ is

$$\delta_1\delta_2\left\{\left(\tfrac{1}{2}a_1\lambda^2 + c\lambda - r_1 - \theta q_1\right)\left(\tfrac{1}{2}a_2\lambda^2 + c\lambda - r_2 - \theta q_2\right) - \theta^2 q_1 q_2\right\},$$

it is immediate from the fact that $r_i$, $\theta q_i > 0$, $i = 1, 2$, that there are two positive and two negative real roots, counting algebraic multiplicity for all positive values of $c$ and $\theta$. The only positive eigenvalue with both

$$\tfrac{1}{2}a_1\lambda^2 + c\lambda - r_1 - \theta q_1 < 0 \text{ and } \tfrac{1}{2}a_2\lambda^2 + c\lambda - r_2 - \theta q_2 < 0$$

is the one closer to zero, and it is therefore the only positive eigenvalue with a positive eigenvector. Thus $K(S)$ has exactly one unstable monotone eigenvalue for all $c > 0$ and $\theta > 0$. Let us denote this eigenvalue throughout by $\underline{\lambda} > 0$ and the first two components of the corresponding eigenvector by $(\underline{v}_1, \underline{v}_2) = \underline{v}$. Let the other positive eigenvalue be $\overline{\lambda} > \underline{\lambda} > 0$ with eigenvector $\overline{v} = (\overline{v}_1, \overline{v}_2)$, $\overline{v}_1 > 0 > \overline{v}_2$. Let

$$TM_S = \operatorname{span}\{(\overline{v}_1, \overline{v}_2, \overline{\lambda}\overline{v}_1, \overline{\lambda}\overline{v}_2), (\underline{v}_1, \underline{v}_2, \underline{\lambda}\underline{v}_1, \underline{\lambda}\underline{v}_2)\} \subset \mathbb{R}^4.$$

Then $TM_S$ is the tangent space to the unstable manifold $M_S$ of (1.3) at $S$. Note that, since $\operatorname{span}\{\overline{v}, \underline{v}\} = \mathbb{R}^2$, this description of $TM_S$ yields a natural one-to-one correspondence $\pi$ from a neighbourhood of the origin in $w$-space onto a neighbourhood of $S$ on $TM_S$ given by

$$\pi(\alpha\overline{v} + \beta\underline{v}) = (\alpha\overline{v}_1 + \beta\underline{v}_1, \alpha\overline{v}_2 + \beta\underline{v}_2, \alpha\overline{\lambda}\overline{v}_1 + \beta\underline{\lambda}\underline{v}_1, \alpha\overline{\lambda}\overline{v}_2 + \beta\underline{\lambda}\underline{v}_2)).$$

Let $\Pi(\alpha, \beta)$ denote the corresponding point on $M_S$ obtained by composing $\pi$ with a chart map from $TM_S$ to $M_S$ whose derivative at $S$ is the identity.

The following result follows almost immediately from the definitions and some standard theory.

(3.4) **Lemma.** *Let $\epsilon > 0$. There exists $\delta > 0$, $k > 0$ and $\tau > 0$ such that if $0 < \beta \leqslant \delta$, $0 \leqslant |\alpha| \leqslant k\beta$ and $w$ is a solution of (1.3) with $(w(0), w'(0)) = \Pi(\alpha, \beta)$ then $w(0) > 0$, $w'(0) > 0$ and* (i) *$w(x) \to S$ as $x \to -\infty$ and $w'(x) \geqslant 0$, $x \in (-\infty, 0]$;* (ii) *if $\alpha = k\delta$ and $\beta = \delta$ then $w_1'(x) \geqslant 0$, $x \in [0, \tau]$, and $w_2(s) < 0$ for some $s \in [0, \tau]$;* (iii) *if $\alpha = -k\delta$ and $\beta = \delta$ then $w_2'(x) \geqslant 0$, $x \in [0, \tau]$ and $w_1(s) < 0$ for some $s \in [0, \tau]$;* (iv) *$\|w(x)\| \leqslant \epsilon$ for all $x \in [0, \tau]$.*





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*Proof.* Let $(w(0), w'(0)) = \Pi(\alpha,\beta)$. Then $w_i(0) = \alpha\overline{v}_i + \beta\underline{v}_i + o((\alpha,\beta))$ and $w_i'(0) = \alpha\overline{\lambda}\overline{v}_i + \beta\underline{\lambda}\underline{v}_i + o((\alpha,\beta))$, $i = 1, 2$, as $(\alpha,\beta) \to 0$. Since $\underline{v} > 0$ there exists $k > 0$ and $\delta > 0$, such that if $|\alpha| \leqslant k\beta$, $0 < \beta \leqslant \delta$ and $(w(0), w'(0)) = \Pi(\alpha,\beta)$ then $w(0) > 0$ and $w'(0) > 0$. Since $\overline{\lambda} > \underline{\lambda} > 0$ it is now immediate by the Stable Manifold Theorem (see Coddington & Levinson 1955, ch. 13, Theorem 4.5, for an explicit statement of the result needed here) that, for such $(\alpha,\beta)$, $w'(x) \geqslant 0$, $x \in (-\infty, 0]$. Also $(w(x), w'(x)) \to (S, 0)$ as $x \to -\infty$ since $\Pi(\alpha,\beta) \in M_S$.

Now let $\tau > 0$ be fixed. Then $w(x) = \alpha e^{\overline{\lambda}x}\overline{v} + \beta e^{\underline{\lambda}x}\underline{v} + o((\alpha,\beta)), x \in [0,\tau]$ as $(\alpha,\beta) \to (0,0)$. (See, for example, Chow & Hale 1982, ch. 3, Theorem 6.2.) If $\alpha = k\delta$, $\beta = \delta$, then

$$w(x) = \delta e^{\underline{\lambda}x}\{k e^{(\overline{\lambda}-\underline{\lambda})x}\overline{v} + \underline{v} + o(1)\}, \qquad x \in [0,\tau],$$

and

$$w'(x) = \delta e^{\underline{\lambda}x}\{\overline{\lambda} k e^{(\overline{\lambda}-\underline{\lambda})x}\overline{v} + \underline{\lambda}\underline{v} + o(1)\}, \qquad x \in [0,\tau],$$

as $\delta \to 0$. Since $\overline{v}_2 < 0$ we can choose $\tau = O(\log 1/k)$ such that $w_2(\tau) < 0$ and $w_1'(x) > 0$, $x \in [0,\tau]$ for $\delta > 0$ sufficiently small. Clearly for $\delta > 0$ sufficiently small $\|w(x)\| \leqslant \epsilon$, $x \in [0,\tau]$. This shows (ii) and (iv) in this case. An analogous argument yields (iii) and (iv). This completes the proof. ■

### (*d*) *Approaching the boundary of* $\Sigma$

Henceforth, let $\delta$ and $k$ be fixed by the preceding lemma, and let $\mathcal{M} = \{\Pi(\alpha,\delta) : -k\delta \leqslant \alpha \leqslant k\delta\} \subset M_S$.

(3.5) **Lemma.** *If $\Pi(\hat{\alpha},\delta)$ is a point of $\mathcal{M}$ whose trajectory meets $\partial\Sigma$ in finite time then it exits from $\Sigma$ at that time by Lemma 3.3. The trajectories through $\Pi(\alpha,\delta)$ for $\alpha$ sufficiently close to $\hat{\alpha}$ also leave $\Sigma$ in finite time and both the exit time and the exit position on $\partial\Sigma$ are continuous functions of $\alpha$ in a neighbourhood of $\hat{\alpha}$.*

*Proof.* This result is a standard consequence of the the fact that the orbit through $\Pi(\hat{\alpha},\delta)$ immediately leaves $\Sigma$ when it first meets $\partial\Sigma$ and classical continuous dependence theory for initial value problems. ■

Let $\mathcal{M}_T$ be the set of points of $\mathcal{M}$ for which the solution of (1.3) with that initial data is monotone and convergent to the equilibrium $T$ as $x \to \infty$. If $E_\pm$ exist, define $\mathcal{M}_{E_\pm}$ similarly.

(3.6) **Lemma.** *The sets $\mathcal{M}_T, \mathcal{M}_{E_+}$ and $\mathcal{M}_{E_-}$ are closed (possibly empty).*

*Proof.* Let $\{(w_n, w_n')\}$ be a sequence in $\mathcal{M}_T$. Without loss of generality we may suppose, using the Ascoli–Arzela theorem and a standard diagonalization argument if necessary, that $\{w_n(x)\}$ converges uniformly on compact intervals to a solution $w(x)$ of (1.3). Also all the sequences $\{w_n^{(k)}(x)\}$ of $k$th derivatives converge uniformly on compact intervals. In particular $w'(x) \geqslant 0$ for $x \geqslant 0$. Since $\{w_n(x) : n \in \mathbb{N},\ x \in [0,\infty)\} \subset (T - \mathbb{R}^2_+)$, by the monotonicity of orbits, it follows that $\{w(x) : x \in [0,\infty)\} \subset (T - \mathbb{R}^2_+)$. Since $w$ is a bounded monotone solution of (1.3) it is now immediate that $(w(x), w'(x))$ converges as $x \to \infty$ to an equilibrium $(w,0)$ of (1.3) with $w \in (T - \mathbb{R}^2_+)$. But even when they exist neither $E_+$ nor $E_-$ is commensurate with $T$ with respect to the partial ordering on $\mathbb{R}^2$ induced by $\mathbb{R}^2_+$ (except of course when $E_-$ and $T$ coincide and $\theta = \theta_0$). Since there are no other non-zero equilibria of (1.3), $w(x) \to T$ as $x \to \infty$ and the proof that $\mathcal{M}_T$ is closed is complete.

Since no pair from $E_+, E_-$ and $T$ are commensurate with respect to the usual partial ordering when they are distinct, the above argument may be repeated to show that $\mathcal{M}_{E_+}$ and $\mathcal{M}_{E_-}$ are closed as well. ■

### (e) *The shooting argument for existence*

We will denote the set $\{T, E_+, E_-\}$ by the set $\{E_1, E_2, E_3\}$ in the notation preceding Lemma 3.3. (We do not exclude the case when two of the these coincide or $E_+$ and $E_-$ do not exist.)

In what follows, we deal with the most awkward case, namely that in which $c \geqslant \max\{c_+(\theta), c_-(\theta), c(\theta)\}$ and the possibility of monotonic travelling waves to each of $E_+$, $E_-$ and $T$ arises. Other cases, when there are fewer than three equilibria with stable monotone eigenvalues, may be treated in exactly the same way. The only difference is that if for $i \in \{1,2,3\}$, $E_i$ exists but does not have a stable monotone eigenvalue, then it cannot be the limit point of a monotone orbit from $S$. This observation enables the shooting argument to be used, essentially without change, since such equilibria play no distinguished role and are treated as any other point of $\partial\Sigma$ from the point of view of continuity of exit times.

We will show, for $i = 1,2,3$, the existence of $\Pi(\alpha,\delta) \in \mathcal{M}$ whose orbit converges monotonically to $E_i$ as $x \to \infty$, and the proof of existence will be complete.

By Lemma 3.5, the time for an orbit through a point of $\mathcal{M}$ to leave $\Sigma$ is continuous at a point $\Pi(\alpha,\delta)$ on $\mathcal{M}$ whose orbit leaves $\Sigma$ in finite time. Hence the exit point on $\partial\Sigma$ is a continuous function of $\alpha$ when $(w(0), w'(0)) = \Pi(\alpha,\delta)$, $-k\delta \leqslant \alpha \leqslant k\delta$, at points $\alpha$ whose orbit leaves $\Sigma$ in finite time.

Let $\alpha_3 = \sup\{\tilde{\alpha}$: the orbit through $\Pi(\alpha,\delta)$ leaves $\Sigma$ at the point of $OE_3$ for all $\alpha \in [-k\delta, \tilde{\alpha}]\}$. (Here $OE_3$ is the open segment of $\partial\Sigma$ joining $O$ and $E_3$ which does not contain $E_1$ or $E_2$.) By Lemma 3.4(iii) the set defining $\alpha_3$ contains $-k\delta$ and so $\alpha_3$ is well defined and $\alpha_3 < k\delta$ by Lemma 3.4(ii). Suppose that the orbit through $\Pi(\alpha_3,\delta)$ is not monotonically convergent to $E_3$ as $x \to \infty$. Then, by Lemmas 3.1 and 3.3, we conclude that either the orbit through $\Pi(\alpha_3,\delta)$ leaves $\Sigma$ at a point not in $\overline{OE_3}$ or it is monotonically convergent to one of $E_1$ or $E_2$. The former cannot be the case by continuity of the exit point as a function of $\alpha \in [-k\delta, k\delta]$ by Lemma 3.5. We conclude that the orbit through $\Pi(\alpha_3,\delta)$ converges monotonically to one of $E_1$ (or $E_2$) as $x \to \infty$. (When there is only one equilibrium the proof is now complete.) If the corresponding solution of (1.3) is denoted by $\hat{w}$ then $\hat{w}(\hat{x})$ is then in a neighbourhood $\hat{N}$ of $E_1$ (or $E_2$) for all $\hat{x}$ sufficiently large. Choose an open neighbourhood $N_3$ of $E_1$ (or $E_2$) which does not intersect $\omega_3$ and let $\hat{x}$ be sufficiently large that $\hat{w}(x) \in N_3$, $\hat{w}'(x) > 0$, for all $x \geqslant \hat{x}$. If $w$ is the solution with $(w(0), w'(0)) = \Pi(\alpha,\delta)$ for $\alpha < \alpha_3$ sufficiently close to $\alpha_3$, then $w(\hat{x}) \in N_3$, and $w'(\hat{x}) > 0$, by continuity. In order for $w$ to leave $\Sigma$ at a point of $OE_3$, as it must do, it is necessary that $w_1$ has a local maximum at some point $\tilde{x} > \hat{x}$ before it meets $\partial\Sigma$. But this is impossible since $w(\tilde{x})$ would then be in $\omega_3$ and $w'(x) > 0$ for all $t \in (\tilde{x}, \hat{x})$. (For $w(x)$ to get to $\omega_3$, $w$ must have a local maximum for $t > \hat{x}$ with $w(x) \notin \omega_3$. See figure 6.) This contradiction shows that the solution with $(w(0), w'(0)) = \Pi(\hat{\alpha},\delta)$ converges monotonically to $E_3$ as $x \to \infty$.

An identical argument yields the existence of a point $\Pi(\alpha_1,\delta)$ such that $w(x) \to E_1$ as $x \to \infty$ when $(w(0), w'(0)) = \Pi(\alpha_1,\delta)$.

Let $\alpha_1$ denote the smallest $\alpha$ in $[-k\delta, k\delta]$ such that $w(x) \to E_1$ as $x \to \infty$ if $(w(0), w'(0)) = \Pi(\alpha_1,\delta)$ and let $\alpha_3$ be the largest $\alpha$ in $[-k\delta, \alpha_1]$ such that

$w(x) \to E_3$ as $x \to \infty$ if $(w(0), w'(0)) = \Pi(\alpha_3, \delta)$. These exist by Lemma 3.6 and the preceding discussion.

Suppose that there is no orbit through $\Pi(\alpha, \delta)$, $\alpha \in (\alpha_1, \alpha_3)$ at $t = 0$ with $w(x) \to E_2$ as $x \to \infty$. A repeat of the argument in the first part of the proof shows that all such orbits must meet $\partial\Sigma$ at a point of the open boundary segment $E_1E_3$. (If an orbit through $\Pi(\alpha, \delta)$ meets $\partial\Sigma$ at $E_1$, $E_3$, or points of $OE_1$ or $OE_3$ then, by Lemma 3.1 and the preceding demonstration, the definition of $\alpha_3$ and $\alpha_1$ would be violated by the existence of another monotone orbit converging to $E_1$ or $E_3$ with $(w(0), w'(0)) = \Pi(\alpha, \delta)$, $\alpha \in (\alpha_1, \alpha_3)$.) Moreover all such orbits must be monotone by Lemma 3.3.

Now if $\alpha \in (\alpha_1, \alpha_3)$ is close to $\alpha_1$ $(\alpha_3)$ then it follows easily from their monotonicity that orbits through $\Pi(\alpha, \delta)$ must meet $\partial\Sigma$ at a point close to $E_1$ $(E_3)$. However if all orbits for $\alpha \in (\alpha_1, \alpha_3)$ meet the boundary segment $E_1E_3$ with non-zero deriative then the continuity of the exit position is guaranteed and an elementary connectedness and continuity argument gives the existence of $\alpha_2 \in (\alpha_1, \alpha_3)$ whose monotone orbit meets $\partial\Sigma$ at $E_2$. But Lemma 3.1 implies that it must do so with zero velocity. Hence the existence of all three monotone orbits is established. ∎

### (f) *Proof of uniqueness modulo translation*

To establish uniqueness of travelling waves from $S$ to $E_i$, $i = 1, 2, 3$, it suffices, because of the change of variables at (1.26), to prove that when $c \geqslant c(\theta)$ then the wave from $S$ to $T$ is unique.

Henceforth suppose that $c \geqslant c(\theta)$. Then there are two stable monotone eigenvalues of $T$, $\lambda_1 \leqslant \lambda_2 < 0$, say, and there are two other eigenvalues $\lambda_3 < \lambda_1$ and $\lambda_4 > \lambda_2$. Let the eigenvectors corresponding to $\lambda_i$ be $e_i$. If $\lambda_1 = \lambda_2$ let the unique normalised eigenvector be denoted by $e_2$. (Recall Lemma 1.13 that when a monotone eigenvalue has algebraic multiplicity 2, its geometric multiplicity is 1.)

Now suppose that $w$ is a monotone travelling wave from $S$ to $T$. If $w = (1, 1) + u$, as in the proof of Lemma 3.1, the equation for $u$ is

$$\tfrac{1}{2}Au'' + cu' + R(u^2) + (R + \theta Q)u = 0 \quad \text{and} \quad u \leqslant 0 \quad \text{on} \quad \mathbb{R}.$$

Suppose that there are two solutions $u$ and $u^*$ of this equation, both of which are monotone and converge to $(-1, -1)$ as $x \to -\infty$ and

$$u(x) = \alpha_2 e^{\lambda_2 x} e_2 + o(e^{\lambda_2 x}) \quad \text{as} \quad x \to \infty, \quad \alpha_2 \neq 0, \quad \text{if} \quad \lambda_1 \neq \lambda_2, \tag{3.7}$$

and

$$u(x) = \alpha_2 x e^{\lambda_2 x} e_2 + O(e^{\lambda_2 x}) \quad \text{as} \quad x \to \infty, \quad \alpha_2 \neq 0, \quad \text{if} \quad \lambda_1 = \lambda_2, \tag{3.8}$$

and similarly for $u^*$ with $\alpha_2^*$ in place of $\alpha_2$. Now observe that the set of solutions of (1.3) is translation invariant. Hence we can replace $u^*$ by $\hat{u}$ where

$$\hat{u}(x) = u^*(x + (\lambda_2)^{-1} \log(\alpha_2/\alpha_2^*)).$$

If

$$h_i(x) = \frac{\hat{u}_i(x)}{u_i(x)}, \text{ then} \quad h_i(x) \to 1 \quad \text{as} \quad |x| \to \infty, \quad i = 1, 2.$$





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Multiplying the equation for $u_i$ by $\hat{u}_i$ and vice versa and subtracting gives

$$\tfrac{1}{2}a_i(u_i\hat{u}_i'' - u_i''\hat{u}_i) + c(u_i\hat{u}_i' - u_i'\hat{u}_i) + \theta q_i(u_i\hat{u}_j - \hat{u}_i u_j) + r_i u_i\hat{u}_i(\hat{u}_i - u_i) = 0.$$

Now suppose that $\sup\{h_1(x) : x \in \mathbb{R}\} = M > 1$. Then clearly it is attained at a point where

$$\hat{u}_1 < u_1 < 0 = u_1\hat{u}_1' - u_1'\hat{u}_1 \geqslant u_1\hat{u}_1'' - u_1''\hat{u}_1,$$

whence

$$u_1\hat{u}_2 > \hat{u}_1 u_2.$$

Therefore $\sup\{h_1(x) : x \in \mathbb{R}\} < \sup\{h_2(x) : x \in \mathbb{R}\}$. But now the argument may be repeated interchanging the subscripts to prove that the opposite strict inequality holds. We conclude that $M = 1$ and the uniqueness result is immediate.

*Proofs of asymptotic expressions.* To complete the proof, we prove that $u$ and $u^*$ must have the asymptotic forms given at (3.7) and (3.8).

We have seen in Lemma 3.2 that $(u, u')$ converges to 0 exponentially fast as $x \to \infty$. Therefore, by standard theory (Coddington & Levinson 1955, ch. 13, Theorem 4.3), we can state that

$$\frac{\log \| (u, u') \|}{x} \to \beta \quad \text{as} \quad x \to \infty$$

where $\beta \in \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ is negative. Recall that $\lambda_1$ and $\lambda_2$ are the only stable monotone eigenvalues and that in our notation $\lambda_3 < \lambda_1 \leqslant \lambda_2 < \lambda_4$. If $\beta = \lambda_4$ then by further standard theory (Coddington & Levinson 1955, ch. 13, Theorem 4.5), it follows that $u(x) = \alpha_4 e^{\lambda_4 x} e_4 + o(e^{\lambda_4 x})$ as $x \to \infty$ for some $\alpha_4 \neq 0$. This is impossible since the eigenvector $e_4$ corresponding to $\lambda_4$ is not a positive vector and $u < 0$. Thus $\beta \leqslant \alpha_2$.

Now we show that $\beta = \lambda_2$. As in the proof of Lemma 3.1, let

$$\phi(x) = e^{-\lambda_2 x}\tilde{v}, \quad \tilde{v} > 0,$$

be the solution of the adjoint problem

$$\tfrac{1}{2}A\phi'' - c\phi' + (R + \theta Q^{\mathrm{T}})\phi = 0 \quad \text{on} \quad \mathbb{R}.$$

If the equation for $u$ is multiplied by $\phi$ and integrated over $(-\infty, x)$ we find that

$$\tfrac{1}{2}\{\langle Au', \tilde{v}\rangle + \lambda_2\langle Au, \tilde{v}\rangle + 2c\langle u, \tilde{v}\rangle\}(x) + e^{\lambda_2 x}\int_{-\infty}^{x} \langle Ru^2, \phi\rangle \, ds = 0.$$

It is immediate that $(u, u')$ does not decay to zero at an exponential rate which is faster than $e^{\lambda_2 x}$ as $x \to \infty$. Thus $\beta = \lambda_2$. In the case $\lambda_1 < \lambda_2$, the proof is complete.

Now suppose that $c = c(\theta)$, $\lambda_1 = \lambda_2$ and note that $e_2$ belongs to the kernel and $\lambda_2 A e_2 + c e_2$ belongs to the range of the operator $\frac{1}{2}\lambda_2^2 A + c\lambda_2 + R + \theta Q$, because the algebraic multiplicity of $\lambda_2$ as an eigenvalue of the four-dimensional stability matrix $K(T)$ is 2, while its geometric multiplicity is 1, in this case. Since $\tilde{v}$ lies in the kernel of the transposed operator it is immediate by the Fredholm Alternative that $\langle(\lambda_2 A + c)e_2, \tilde{v}\rangle = 0$. Now suppose, seeking a contradiction (see Coddington & Levinson 1955, ch. 13, Theorem 4.5), that

$$u(x) = \alpha e^{\lambda_2 x} e_2 + o(e^{\lambda_2 x}) \quad \text{and} \quad u'(x) = \alpha\lambda_2 e^{\lambda_2 x} e_2 + o(e^{\lambda_2 x}) \quad \text{as} \quad x \to \infty.$$





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When this is substituted into the integral identity above we find that

$$o(\mathrm{e}^{\lambda_2 x}) = \mathrm{e}^{\lambda_2 x} \int_{-\infty}^{x} \langle Ru^2, \phi \rangle \, \mathrm{d}s.$$

This is a contradiction which shows that in this case $u(x) = \alpha x \mathrm{e}^{\lambda_2 x} + O(\mathrm{e}^{\lambda_2 x})$ as $x \to \infty$ for some non-zero $\alpha$. This completes the proof. ∎

## 4. Proofs of probabilistic results

### (a) A one-particle model

Let $I := \{1, 2\}$ and $E := \mathbb{R} \times I$. We consider a process $(\xi, \eta)$ on $E$, where $\eta$ is an autonomous Markov chain with $Q$-matrix $\theta Q$, and where, while $\eta = y \in I$, $\xi$ is a Brownian motion with zero drift and with constant variance coefficient $a(y)$. Thus, $(\xi, \eta)$ has formal generator $\mathcal{H}$, where

$$(\mathcal{H}F)(x, y) = \theta \sum_{j \in I} Q(y, j) F(x, j) + \tfrac{1}{2} a(y) \frac{\partial^2 F}{\partial x^2}.$$

We write $\mathbb{P}_{x,y}$ (with associated expectation $\mathbb{E}_{x,y}$) for the law of our process when it starts from one particle at position $\xi(0) = x$ and of type $\eta(0) = y$. By martingale (respectively, local martingale, supermartingale,...) we mean a process which is for every $\mathbb{P}_{x,y}$ a martingale (respectively,...) relative to the natural filtration ($\mathbb{P}_{x,y}$-augmented if you wish) of the $(\xi, \eta)$ process. Let $r$ be a positive function on $I$.

Let $\lambda < 0$, let $\gamma_\lambda$ be the Perron–Frobenius eigenvalue of $\frac{1}{2}\lambda^2 A + \theta Q + R$, and let $v_\lambda$ be the associated eigenvector with $v_\lambda(1) = 1$. Set

$$\zeta_\lambda(t) := \exp\left( \int_0^t r(\eta_s) \, \mathrm{d}s \right) v_\lambda(\eta_t) \exp(\lambda \xi_t - \gamma_\lambda t). \tag{4.1}$$

Itô's formula shows that $\zeta_\lambda$ is a local martingale; and since $\zeta_\lambda$ is also non-negative, it is a supermartingale. It is important that $\zeta_\lambda$ is in fact a true martingale. To see this, first pick $\mu < \lambda \, (< 0)$. Then apply the standard inequality for non-negative supermartingales to $\zeta_\mu$ to obtain for $\alpha > 0$,

$$\mathbb{P}_{x,y}\left( \sup_{s \leqslant t} \zeta_\mu(s) \geqslant \alpha \right) \leqslant \alpha^{-1} \mathbb{E}_{x,y}\left(\zeta_\mu(0)\right) = \alpha^{-1} v_\mu(y) \mathrm{e}^{\mu x},$$

from which it follows that, for $u > 0$,

$$\mathbb{P}_{x,y}\left( \inf_{s \leqslant t} \xi_s \leqslant -u \right) \leqslant K_1(x, y, t) \mathrm{e}^{\mu u}$$

for some finite $K_1(x, y, t)$. However, it is also true that, for some finite $K_2(x, y, t)$,

$$\sup_{s \leqslant t} \zeta_\lambda(s) \leqslant K_2(x, y, t) \exp\left( \lambda \inf_{s \leqslant t} \xi_s \right).$$

Hence $\sup_{s \leqslant t} \zeta_\lambda(s)$ is in each $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P}_{x,y})$, and hence $\zeta_\lambda$ is a true martingale.

We let $(U_\lambda : \lambda > 0)$ be the resolvent of the $(\xi, \eta)$ process. We consider $U_\lambda$ as a bounded operator of norm $\lambda^{-1}$ on the space $\mathrm{b}\mathcal{B}$ of bounded Borel functions on $\mathbb{R}$.

### (*b*) *Martingales for the multitype process*

We now study the multitype process at (1.33) of § 1 *f*. The state-space for this process is

$$S := \bigcup_{n\geqslant 1} (\{n\} \times \mathbb{R}^n \times I^n).$$

We give the formula for the formal generator $\mathcal{G}$ of the process at (1.33), in case it clarifies the structure for someone who likes generators. We have

$$\mathcal{G} = \mathcal{G}_A + \mathcal{G}_Q + \mathcal{G}_R, \tag{4.2}$$

where, for $n \geqslant 1$, $x \in \mathbb{R}^n$, and $y \in I^n$, we have (for $F : S \to \mathbb{R}$)

$$(\mathcal{G}_A F)(n;x;y) = \sum_{k=1}^{n} \tfrac{1}{2} a(y_k) \frac{\partial^2 F}{\partial x_k^2},$$

$$(\mathcal{G}_Q F)(n;x;y) = \theta \sum_{k=1}^{n} \sum_{j \neq y_k} Q(y_k, j)\{F(n;x;s_{k,j}(y)) - F(n;x;y)\},$$

$$(\mathcal{G}_R F)(n;x;y) = \sum_{k=1}^{n} r(y_k)\{F(n+1;(x,x_k);(y,y_k)) - F(n;x;y)\},$$

where $s_{k,j}(y) := (y_1, \dots, y_{k-1}, j, y_{k+1}, \dots, y_n)$ and $(x, x_k) := (x_1, \dots, x_n, x_k)$, etc. If $F : [0,\infty) \times S \to \mathbb{R}$ and

$$\left\{\left(\frac{\partial}{\partial t} + \mathcal{G}\right) F\right\}(t;n;x;y) = 0 \qquad (n \geqslant 1,\ x \in \mathbb{R}^n,\ y \in I^n), \tag{4.3}$$

then $F(t;N(t);X(t);Y(t))$ is a local martingale.

In particular, if $u$ solves our coupled reaction-diffusion equation (1.2), then, for $t > 0$,

$$M(r) := \prod_{k=1}^{N(r)} u(t-r; X_k(r); Y_k(r)) \tag{4.4}$$

defines a 'multiplicative' local martingale $M$ on time-parameter set $[0,t]$. If $0 \leqslant u \leqslant 1$, then $0 \leqslant M \leqslant 1$, so that $M$ is a true martingale and

$$\mathbb{E}_{x,y} M(0) = \mathbb{E}_{x,y} M(t),$$

that is,

$$u(t,x,y) = \mathbb{E}_{x,y} \prod_{k=1}^{N(t)} f(X_k(t), Y_k(t)) = \mathbb{E}_{0,y} \prod_{k=1}^{N(t)} f(x + X_k(t), Y_k(t)), \tag{4.5}$$

where $f(x,y) = u(0,x,y)$. Part (i) of Theorem 1.36 is proved.

Still guided by McKean, we now look at 'additive' martingales. If $h : [0,\infty) \times \mathbb{R} \times I \to \mathbb{R}$ satisfies the *linear* equation

$$\frac{\partial h}{\partial t} + \tfrac{1}{2} A \frac{\partial^2 h}{\partial x^2} + \theta Q h + R h = 0,$$

then, again by using (4.3),

$$\sum_{k=1}^{N(t)} h(t, X_k(t), Y_k(t)) \text{ is a local martingale.}$$

Note that Part (ii) of Theorem 1.36 follows from the fact that $M$ at (4.4) is a local martingale and from the local-martingale property just established.

We now recall the definition of $Z_\lambda$ from (1.40). Let $c > c(\theta)$. Let $\lambda$ be the monotone eigenvalue of $K_{c,\theta}(T)$ nearer to 0. Thus $-\lambda c$ is the Perron–Frobenius eigenvalue of $\frac{1}{2}A\lambda^2 + \theta Q + R$; let $v_\lambda$ be the corresponding eigenvector with $v_\lambda(1) = 1$. Define

$$(4.6) \qquad Z_\lambda(t) := \sum_{k=1}^{N(t)} v_\lambda(Y_k(t)) \exp\{\lambda[X_k(t) + ct]\}.$$

Then $Z_\lambda$ is an 'additive' local martingale; and since it is non-negative, it is also a supermartingale. We wish to show that $Z_\lambda$ is a true martingale. For this purpose, we need the following intuitively obvious lemma.

We now remove the 'source' and 'target' connotations from $S$ and $T$, leaving them free to denote stopping times. We write $K_{c,\theta}$ instead of $K_{c,\theta}(T)$.

(4.7) **Lemma.** *For any non-negative Borel function $f$ on $\mathbb{R} \times I$, we have*

$$\mathbb{E}_{x,y} \sum_{k=1}^{N(t)} f(X_k(t), Y_k(t)) = \mathbb{E}_{x,y} \exp\left\{\int_0^t r(\eta_s)\, \mathrm{d}s\right\} f(\xi_t, \eta_t).$$

*Proof.* It is enough to prove this result for $f$ in $C_0$. It is further enough to show that if $\mu > \max(r_1, r_2)$ and $S$ is a exponentially distributed random variable of rate $\mu$, independent of our branching process and of $(\xi, \eta)$, then, for $f \in C_0$,

$$(4.8) \quad g(x,y) := \mathbb{E}_{x,y} \sum_{k=1}^{N(S)} f\left(X_k(S), Y_k(S)\right) = \mathbb{E}_{x,y} \exp\left\{\int_0^S r(\eta_t)\, \mathrm{d}t\right\} f(\xi_S, \eta_S).$$

The finiteness of $g$ is clear. Let $T$ be the first branch-time of the $(N, X, Y)$ process. On splitting $g$ according to which of $S$ and $T$ is the smaller, we find that

$$g(x,y) := g^*(x,y) + 2g^{**}(x,y),$$

where $g^*$ and $g^{**}$ are the bounded Borel functions:

$$\begin{aligned} g^* &:= \mathbb{E}_{..}[f(X_1(S), Y_1(S)); S < T] \\ &= \mathbb{E}_{..} f(X_1(S), Y_1(S)) \exp\left(-\int_0^S r(Y_1(t))\, \mathrm{d}t\right); \\ g^{**} &:= \mathbb{E}_{..}[g(X_1(T), Y_1(T)); S > T] = \mathbb{E}_{..} \mathrm{e}^{-\mu T} g(X_1(T), Y_1(T)). \end{aligned}$$

Formally, the Feynman–Kac formula shows that

$$(\mu + R - \mathcal{H})g^* = \mu f,$$

and this suggests the rigorous formulae (see Williams 1979, § III.39):

$$(I + U_\mu R)g^* = \mu U_\mu f, \qquad g^* = \mu(I + U_\mu R)^{-1} U_\mu f.$$




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(Note that, since $\mu > \max\{r(i) : i \in I\}$, we have $\|U_\mu R\| < 1$.)

Next, we use the fact that

$$I_{\{T \leqslant t\}} - \int_0^{T\wedge t} r(Y_1(s))\,\mathrm{d}s \text{ is a martingale.}$$

We integrate the bounded previsible process $\mathrm{e}^{-\mu t} g((X_1(t), Y_1(t-))$ against this martingale, and use the fact that $Y(T-) = Y(T)$ almost surely, to obtain

$$\begin{aligned} g^{**} = Bg :&= \mathbb{E}_{..} \int_0^T \mathrm{e}^{-\mu t} r(Y_1(t)) g(X_1(t), Y_1(t))\,\mathrm{d}t \\ &= \mathbb{E}_{..} \int_0^\infty - \mathbb{E}_{..} \int_T^\infty \\ &= (U_\mu R g) - \mathbb{E}_{..} \mathrm{e}^{-\mu T} (U_\mu R g)(X_1(T), Y_1(T)) \\ &= U_\mu R g - B U_\mu R g. \end{aligned}$$

Thus, on b$\mathcal{B}$,

$$B(I + U_\mu R) = U_\mu R, \qquad B = U_\mu R (I + U_\mu R)^{-1} = (I + U_\mu R)^{-1} U_\mu R.$$

We now have

$$g = (I + U_\mu R)^{-1} \{\mu U_\mu f + 2 U_\mu R g\},$$

so that

$$(I - U_\mu R) g = \mu U_\mu f, \qquad g = \mu (I - U_\mu R)^{-1} U_\mu f.$$

The Feynman–Kac formula gives the same expression for the right-hand side of (4.8). ■

We know that $Z_\lambda$ is a supermartingale. From Lemma 4.7 and the martingale property of the process $\zeta_\lambda$ at (4.1), the expectation of $Z_\lambda(t)$ is constant in $t$. Hence, $Z_\lambda$ is a true martingale.

### (*c*) *Convergence properties of $Z_\lambda$ martingales*

The full assumptions which we have made about $Z_\lambda$ (that $c > c(\theta)$, that $\lambda$ is the monotone eigenvalue of $K_{c,\theta}$ nearer to 0, etc.) will now be needed in proving that $Z_\lambda$ converges in $\mathcal{L}^p$ for some $p > 1$ (and hence in $\mathcal{L}^1$). According to one of Doob's theorems, we need only show that $Z_\lambda$ is bounded in $\mathcal{L}^p$ for some $p > 1$.

For investigating convergence in $\mathcal{L}^p$, the following lemma is indispensable. The result is taken from Neveu (1987), and the method of using it is adapted from that paper.

(4.9) **Lemma (Neveu).** *Let $p \in (1, 2]$. For any finite sequence $W_1, \dots, W_n$ of non-negative independent variables in $\mathcal{L}^p$ and any sequence $c_1, \dots, c_n$ of non-negative real numbers, we have*

$$\psi\left(\sum_{k=1}^n c_k W_k\right) \leqslant \sum_{k=1}^n c_k^p \psi(W_k),$$

*where $\psi(W) := \mathbb{E}(W^p) - \mathbb{E}(W)^p$ for $W \in \mathcal{L}^p$.*

*Proof of $\mathcal{L}^1$ convergence of $Z_\lambda$.* Fix $t > 0$. Because of the branching character of the $(N, X, Y)$ process, we have for each $s > 0$,

$$Z_\lambda(s+t) = \sum_{k=1}^{N(s)} \exp\{\lambda[X_k(s) + cs]\} W_k(t, s),$$

where, conditionally on $\mathcal{F}_s$, the $W_k(t,s)$ are independent, each with the $\mathbb{P}_{0,y(k)}$ law of $Z_\lambda(t)$ where $y(k) = Y_k(s) \in I$. Since $t$ is fixed, and $I$ is finite, Neveu's lemma applied conditionally on $\mathcal{F}_s$ gives

$$\mathbb{E}_{x,y}\{Z_\lambda(s+t)^p \mid \mathcal{F}_s\} - Z_\lambda(s)^p \leqslant K_1(t,x,\lambda) \sum_{k=1}^{N(s)} \exp\{\lambda p[X_k(s) + cs]\},$$

so that, on taking expectations,

$$\mathbb{E}_{x,y}\{Z_\lambda(s+t)^p\} - \mathbb{E}_{x,y}\{Z_\lambda(s)^p\} \leqslant K_1(t,x,\lambda)\mathbb{E}_{x,y} \sum_{k=1}^{N(s)} \exp\{\mu[X_k(s) + cs]\},$$

where $\mu := \lambda p < \lambda < 0$. We must choose $p$ in $(1, 2]$ sufficiently close to 1 that (see Lemma 2.4) $c > c_1$, where we write $-\mu c_1$ for the Perron–Frobenius eigenvalue of $\frac{1}{2}\mu^2 A + \theta Q + R$ and $v_\mu$ for the corresponding eigenvector with $v_\mu = 1$. But then

$$\begin{aligned}\mathbb{E}_{x,y} &\sum_{k=1}^{N(s)} \exp\{\mu[X_k(s) + cs]\} \\ &\leqslant K_2(x,\mu,y)\exp\{\mu(c-c_1)s\}\mathbb{E}_{x,y}\sum_{k=1}^{N(s)} v_\mu(Y_k(s))\exp\{\mu[X_k(s) + c_1 s]\} \\ &\leqslant K_3(x,\mu,y)\exp\{\mu(c-c_1)s\},\end{aligned}$$

since $Z_\mu$ is a martingale. Hence,

$$\begin{aligned}\sum_m \mathbb{E}_{x,y}&\{Z_\lambda(ms+s+t)^p - Z_\lambda(ms+t)^p\} \\ &\leqslant K_4(s,x,\mu,y)\sum_m \exp\{\mu m(c-c_1)s\} < \infty,\end{aligned}$$

and $Z_\lambda$ is bounded in $\mathcal{L}^p$. ■

Now we prove that, with the same notation and assumptions,

(4.10) $$w(y) := \mathbb{P}_{x,y}(Z_\lambda(\infty) = 0) = 0 \text{ for all } (x,y).$$

(The fact that $w(y)$ does not depend on $x$ is obvious.)

*Proof of (4.10).* Let $J$ be the first jump time of $Y_1$ and let $T$ be the first branch time of $(N, X, Y)$. On decomposing $w(y)$ according as $T < J$ or $T > J$, we obtain

$$w(y) = \frac{r(y)w(y)^2 + \theta\sum_{z\neq y} Q(y,z)w(z)}{r(y) + \theta q(y)},$$

so that $Rw = R(w^2) + \theta Q w$. By Lemma 2.7, $w \equiv 0$ on $I$ or $w \equiv 1$ on $I$. When $Z_\lambda$ converges in $\mathcal{L}^1$, then, obviously, $w \equiv 0$ on $I$. ■

*Theorem 1.39 is now proved.*





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For the proof of the uniqueness modulo translation of the monotonic travelling wave from source to target, we need the following result.

(4.11) **Lemma.** *Suppose that $c > c(\theta)$ and that $\beta$ is the stable monotone eigenvalue of $K_{c,\theta}$ further from 0, and that $v_\beta$ is the associated Perron–Frobenius eigenvector of $\frac{1}{2}\beta^2 A + \theta Q + R$ with $v_\beta(1) = 1$. Then, almost surely,*

$$Z_\beta(t) = \sum_{k=1}^{N(t)} v_\beta(Y_k(t)) \exp\{\beta[X_k(t) + ct]\} \to 0$$

*as $t \to \infty$.*

We prove this result too by modifying an argument in Neveu (1987).

*Proof.* Let $0 < p < 1$. Note that therefore, for $u, v > 0$, we have

$$(u + v)^p \leqslant u^p + v^p.$$

Again, let $J$ be the first jump time of $Y_1$ and let $T$ be the first branch time of $(N, X, Y)$. An obvious decomposition of the form

$$Z_\beta(\infty) = \begin{cases} \exp\{\beta[X_1(J) + cJ]\} Z_\beta^{(1)}(\infty) & \text{if } J < T, \\ \exp\{\beta[X_1(T) + cT]\}[Z_\beta^{(2)}(\infty) + Z_\beta^{(3)}(\infty)] & \text{if } T < J, \end{cases}$$

leads to the formula

$$\begin{aligned} g(y) := \mathbb{E}_{0,y}[Z_\beta(\infty)^p] \leqslant \mathbb{E}_{0,y} &\exp\{\alpha[X_1(J) + cJ]\} I_{J<T}\, g(Y_1(J)) \\ &+ 2\mathbb{E}_{0,y} \exp\{\alpha[X_1(T) + cT]\} I_{T<J}\, g(Y_1(T)), \end{aligned}$$

where $\alpha := p\beta$. On evaluating these expectations, we obtain

$$g(y) \leqslant \frac{\left\{\theta \sum_{z \neq y} Q(y,z) g(z)\right\} + 2r(y)g(y)}{[-\frac{1}{2}a(y)\alpha^2 - c\alpha + r(y) + \theta q(y)]},$$

the denominator being positive for $p$ near 1, and this rearranges to give

$$0 \leqslant (\tfrac{1}{2}\alpha^2 A + \alpha c I + \theta Q + R) g.$$

We know that $g \geqslant 0$ on $I$. Lemma 2.5 shows that if we choose $p$ sufficiently close to 1, then $g = 0$. The lemma is proved. ■

### (*d*) *Proof of Theorem 1.49*

We now prove the various steps in Theorem 1.49 which lead to the important result

(4.12) $$\lim t^{-1} L(t) = -c(\theta) \quad \text{(a.s.)} \quad \text{where} \quad L(t) := \inf_{k \leqslant N(t)} X_k(t).$$

*Part* (i). Let $c > c(\theta)$, and let $\lambda$ be the monotone eigenvalue of $K_{c,\theta}$ nearer to 0. Let $Z_\lambda$ be the associated martingale. We see by considering the position of the leftmost particle that

$$Z_\lambda(t) := \sum_{k=1}^{N(t)} v_\lambda(Y_k(t)) \exp\{\lambda[X_k(t) + ct]\} \geqslant \min(v_\lambda(1), v_\lambda(2)) \exp\{\lambda[L(t) + ct]\}.$$

Since $Z_\lambda(\infty)$ exists a.s., $\liminf[L(t)+ct] > -\infty$, a.s., so that

$$\liminf t^{-1}L(t) \geqslant -c = \lambda^{-1}\Lambda_{\mathrm{PF}}(\lambda). \qquad \blacksquare$$

*Part* (ii). Recall that we are now working with the $\mathbb{P} := \mathbb{P}_{0,1}$ law, so that $Z_\lambda(0) = 1$. Since $Z_\lambda$ converges in $\mathcal{L}^1$, we can define a probability measure $Q_\lambda$ on $\mathcal{F}_\infty$ via

$$\mathrm{d}Q_\lambda/\mathrm{d}\mathbb{P} = Z_\lambda(\infty) \text{ on } \mathcal{F}_\infty, \qquad \text{whence} \quad \mathrm{d}Q_\lambda/\mathrm{d}\mathbb{P} = Z_\lambda(t) \text{ on } \mathcal{F}_t.$$

Define

$$M_\lambda(t) := Z_\lambda(t)^{-1}\frac{\partial}{\partial\lambda}Z_\lambda(t).$$

Because $(\partial/\partial\lambda)Z_\lambda(t)$ is a $\mathbb{P}$-martingale, $M_\lambda$ is a $Q_\lambda$-martingale.

For $t \geqslant 0$ and $1 \leqslant k \leqslant N(t)$, define

$$H(t,k) := \frac{v_\lambda(Y_k(t))\exp[\lambda X_k(t) - \Lambda_{\mathrm{PF}}(\lambda)t]}{\sum_{j=1}^{N(t)} v_\lambda(Y_j(t))\exp[\lambda X_j(t) - \Lambda_{\mathrm{PF}}(\lambda)t]}.$$

Note that $H(t,k) \geqslant 0$ and $\sum_j H(t,j) = 1$. We have

$$M_\lambda(t) = \sum_{k=1}^{N(t)} H(t,k)\{u_\lambda(Y_k(t)) + X_k(t) - \Lambda'_{\mathrm{PF}}(\lambda)t\},$$

where $u_\lambda(j) := v'_\lambda(j)/v_\lambda(j)$, so that

$$t^{-1}M_\lambda(t) \geqslant t^{-1}\left\{\min_{i\in I} u_\lambda(i)\right\} + t^{-1}L(t) - \Lambda'_{\mathrm{PF}}(\lambda). \tag{4.13}$$

By Jensen's inequality,

$$M_\lambda(t)^2 \leqslant \sum H(t,k)\{u_\lambda(Y_k(t)) + X_k(t) - \Lambda'_{\mathrm{PF}}(\lambda)t\}^2. \tag{4.14}$$

However,

$$Z_\lambda(t)^{-1}\frac{\partial^2}{\partial\lambda^2}Z_\lambda(t)$$

is a $Q_\lambda$-martingale, and it is easily confirmed that this fact together with (4.14) shows that the $Q_\lambda$-expectation of $M_\lambda(t)^2$ satisfies

$$Q_\lambda[M_\lambda(t)^2] \leqslant K_1(\lambda) + K_2(\lambda)t$$

for finite $K_1(\lambda)$ and $K_2(\lambda)$. Hence, for $\epsilon > 0$,

$$Q_\lambda\left(\sup\{s^{-1}|M_\lambda(s)| : 2^{n-1} \leqslant s \leqslant 2^n\} \geqslant \epsilon\right)$$
$$\leqslant Q_\lambda\left(\sup_{s\leqslant 2^n}|M_\lambda(s)| \geqslant \epsilon 2^{n-1}\right) \leqslant (\epsilon 2^{n-1})^{-2}[K_1(\lambda) + 2^n K_2(\lambda)],$$

by Doob's submartingale inequality. By the Borel–Cantelli lemma, we have

$$t^{-1}M_\lambda(t) \to 0, \quad \text{a.s.},$$

whence, from (4.13),

$$\limsup_{t\to\infty} t^{-1}L(t) \leqslant \Lambda'_{\mathrm{PF}}(\lambda).$$

Part (ii) of Lemma 2.4 clinches Part (iii) of Theorem 1.49, and the proof of (4.12) is complete. $\blacksquare$

### (*e*) *Rounding off the probability*

We have proved the McKean representation

$$u(t,x,y) = \mathbb{E}_{x,y} \prod_{k=1}^{N(t)} u(0, X_k(t), Y_k(t)) \tag{4.15}$$

for the unique solution of our coupled reaction-diffusion equation (1.2) when $0 \leqslant u \leqslant 1$ and the initial data $u(0,\cdot,\cdot)$ are given. Because $t^{-1}L(t) \to -c(\theta)$ (a.s.), the last part of Theorem 1.44 is now obvious.

*Proof of Theorem 1.41.* We take $c > c(\theta)$, let $\lambda$ be the monotone eigenvalue of $K_{c,\theta}$ nearer to 0, and let $v_\lambda$ be the corresponding eigenvector with $v_\lambda(1) = 1$.

We are guided by McKean (1975). Suppose that $u$ solves (1.2), that $0 \leqslant u \leqslant 1$ and that, for all $y$,

$$1 - u(0,r,y) \sim v_\lambda(y)\mathrm{e}^{\lambda r} \qquad (r \to \infty). \tag{4.16}$$

For (temporarily) fixed $\epsilon > 0$, we have for large $r$,

$$\exp\left\{-(1+\epsilon)v_\lambda(y)\mathrm{e}^{\lambda r}\right\} \leqslant u(0,r,y) \leqslant \exp\left\{-(1-\epsilon)v_\lambda(y)\mathrm{e}^{\lambda r}\right\}.$$

Now since $L(t) + ct \to \infty$ (a.s.), we shall (a.s.) have for large $t$,

$$\exp\left\{-(1+\epsilon)Z_\lambda(t)\right\} \leqslant \prod_{k=1}^{N(t)} u(0, X_k(t) + ct, Y_k(t)) \leqslant \exp\left\{-(1-\epsilon)Z_\lambda(t)\right\},$$

so that, with boundedness providing the rigour,

$$\begin{aligned} \mathbb{E}_{x,y} \exp\left\{-(1+\epsilon)Z_\lambda(\infty)\right\} &\leqslant \liminf u(t, x+ct, y) \\ &\leqslant \limsup u(t, x+ct, y) \\ &\leqslant \mathbb{E}_{x,y} \exp\left\{-(1-\epsilon)Z_\lambda(\infty)\right\}. \end{aligned}$$

On letting $\epsilon \downarrow 0$, we now obtain the desired result

$$u(t, x+ct, y) \to w(x,y) := \mathbb{E}_{x,y} \exp\{-Z_\lambda(\infty)\}. \tag{4.17}$$

*Existence of a monotonic travelling wave from $S$ to $T$ when $c > c(\theta)$.* It is now intuitively obvious, and not that difficult to prove directly from the branching property, that the function $w(\cdot,\cdot)$ in (4.17) is a monotonic travelling wave from $S$ to $T$. Note that

$$w(x,y) = \mathbb{E}_{0,y} \exp\{-\mathrm{e}^{\lambda x} Z_\lambda(\infty)\},$$

that $w(x,y) \to 0$ as $x \to -\infty$ because $Z_\lambda > 0$ (a.s.), and that

$$1 - w(x,y) \sim v_\lambda(y)\mathrm{e}^{\lambda x} \qquad (x \to \infty) \tag{4.18}$$

because $Z_\lambda$ converges $\mathcal{L}^1$ whence

$$\mathbb{E}_{x,y} Z_\lambda(\infty) = \mathbb{E}_{x,y} Z_\lambda(0) = \mathrm{e}^{\lambda x} v_\lambda(y).$$

*Uniqueness modulo translation of the monotonic travelling wave from $S$ to $T$.* Let $c > c(\theta)$, and let $\tilde{w}$ be a monotonic travelling wave from $S$ to $T$. We know from differential-equation theory, and have used in § 3, the result that either a suitable translate of $\tilde{w}$ satisfies (4.18), or a suitable translate of $\tilde{w}$ satisfies

$$1 - \tilde{w}(x,y) \sim v_\beta(y)\mathrm{e}^{\beta x} \qquad (x \to \infty), \tag{4.19}$$

where $\beta$ is the monotone eigenvalue of $K_{c,\theta}$ further from 0.

If $\tilde{w}$ satisfies (1.3) and (4.18), then $u(t,x,y) := \tilde{w}(x-ct,y)$ satisfies (1.2) and (4.16), so that from (4.17), we must have $\tilde{w} = w$. If $\tilde{w}$ satisfied (4.19), then we would have

$$\tilde{w}(x,y) = \mathbb{E}_{x,y} \exp\{-Z_\beta(\infty)\} = 1,$$

because $Z_\beta(\infty) = 0$ (a.s.) by Lemma 4.11; and $\tilde{w}$ would not go from $S$ to $T$. ■

The proof of unicity is now complete, and with it the proof of Theorem 1.41. ■

It is interesting to compare the above probabilistic proofs of existence and uniqueness modulo translation with the analytic proofs given in § 3. We could, for example, use the ODE results in § 3 to obtain results on $\mathcal{L}^1$ convergence of our martingales. The probabilistic study of the $c = c(\theta)$ case (following Neveu's work) and of other convergence properties more refined than those considered here will appear in a paper by one of us (J.W.). Of course, the methods in § 3 dealt with the $c = c(\theta)$ case.

*The Doob-h-transform associated with equation (1.26).* For definiteness, we work once more with the $\mathbb{P} := \mathbb{P}_{0,1}$ measure. Suppose that $0 < \theta \leqslant \theta^*$, so that $E_+$ and $E_-$ exist. Let $E_+$ have coordinates $(\alpha_1, \alpha_2)$ in $\mathbb{R}^2$. Suppose that $E_+ \neq (1,1)$. Then

$$M(t) := \alpha_1^{-1} \alpha_1^{N_1(t)} \alpha_2^{N_2(t)},$$

where $N_i(t)$ is the number of particles of type $i$ at time $t$, is a positive martingale of constant expectation 1. We may therefore define a probability measure $\tilde{\mathbb{P}}$ on $\mathcal{F}_\infty$ via the fact that

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = M(t) \text{ on } \mathcal{F}_t.$$

The measure $\tilde{\mathbb{P}}$ is associated with the set-up $(a_1, a_2, \tilde{q}_1, \tilde{q}_2, \tilde{r}_1, \tilde{r}_2, \theta)$ defined at (1.26) in exactly the same way as $\mathbb{P}$ is associated with the original set-up.

It is important to realize that though $\tilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent on every $\mathcal{F}_t$, they are mutually singular on $\mathcal{F}_\infty$. This is because $M(\infty) = 0$ almost surely ($\mathbb{P}$); for if we let $(T_n)$ be the sequence of stopping times at which $N_1(t)$ increases by 1, then $M(T_n) = \alpha_1 M(T_n-)$, so that $M(\infty) = \alpha_1 M(\infty)$.

We thank the referees for their helpful comments. A.C. was supported by an SERC Research Assistantship. S.H. and J.W. were supported by an SERC Research Studentship.

## 5. Addendum

Since the paper was submitted, Crooks (1994) has obtained the necessary generalizations in linear algebra for an extension of our probabilistic proofs of existence and uniqueness to the $n$-type case. We have also recently become aware of some analytic work on the $n$-type case due to Vol'pert & Vol'pert (1993) who use the classical method of upper- and lower-solutions to obtain existence, but not uniqueness. Other models are under investigation, using the methods of this paper.

Forthcoming work by Harris, and by Harris & Williams, proves the conjecture made in § 1 *a*, emphasizing the role of the Legendre transform in turning large-deviation heuristics into precise martingale results.





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## References


Beyn, W.-J. 1990 The numerical computation of connecting orbits in dynamical systems. *IMA J. Numer. Analysis* **9**, 379–405.

Biggins, J. D. 1977 Martingale convergence in the branching random walk. *J. appl. Probab.* **14**, 25–37.

Biggins, J. D. 1979 Growth rates in the branching random walk. *Z. Wahrscheinlichkeitstheorie* **48**, 17–34.

Biggins, J. 1992 Uniform convergence in the branching random walk. *Ann. Probab.* **20**, 137–151.

Bramson, M. D. 1978 Maximal displacement of branching Brownian motion. *Communs pure appl. Math.* **31**, 531–581.

Bramson, M. D. 1983 Convergence of solutions of the Kolmogorov equation to travelling waves. *Mem. Am. math. Soc.* **285**, 1–190.

Britton, N. F. 1986 *Reaction-diffusion equations and their applications to biology.* New York: Academic Press.

Carr, J. 1981 *Applications of centre manifold theory.* Heidelberg and New York: Springer-Verlag.

Chauvin, B. & Rouault, A. 1988 KPP equation and supercritical branching Brownian motion in the subcritical speed area. Application to spatial trees. *Probab. Thy. Rel. Fields* **80**, 299–314.

Chauvin, B. & Rouault, A. 1990 Supercritical branching Brownian motion and K-P-P equation in the critical speed-area. *Math. Nachr.* **149**, 41–59.

Chow, S.-N. & Hale, J. K. 1982 *Methods of bifurcation theory.* Heidelberg and New York: Springer-Verlag.

Coddington, E. A. & Levinson, N. 1955 *Theory of ordinary differential equations.* New York: McGraw-Hill.

Crooks, E. C. M. 1994 Remarks on the Vol'pert theory of travelling-wave solutions for parabolic systems. Preprint, University of Bath.

Deuschel, J.-D. & Stroock, D. W. 1989 *Large deviations.* Boston: Academic Press.

Doedel, E., Keller, H. B. & Kernevez, J. P. 1991*a* Numerical analysis and control of bifurcation problems. I. Bifurcation in finite dimensions. *Int. J. Bifurcation Chaos* **1**, 493–520.

Doedel, E., Keller, H. B. & Kernevez, J. P. 1991*b* Numerical analysis and control of bifurcation problems. II. Bifurcation in infinite dimensions. *Int. J. Bifurcation Chaos* **1**, 745–772.

Dunbar, S. 1984 Traveling wave solutions of diffusive Lotka–Volterra equations: a heteroclinic connection in $\mathbb{R}^4$. *Trans. Am. math. Soc.* **286**, 557–594.

Dunbar, S. 1986 Traveling waves in diffusive predator–prey equations: periodic orbits and point-to-point hetroclinic orbits. *SIAM J. appl. Math.* **46**, 1057–1078.

Ekeland, I. & Temam, R. 1976 *Convex analysis and variational problems.* Amsterdam: North-Holland.

Ellis, R. S. 1985 *Entropy, large deviations, and statistical mechanics.* Heidelberg and New York: Springer-Verlag.

Elworthy, K. D. & Zhao, H. Z. 1993 The travelling wave solutions of scalar generalized KPP equations via classical mechanics and stochastic approaches. (In the press.)

Elworthy, K. D., Truman, A. & Zhao, H. Z. 1993 Approximate travelling waves for generalized KPP equations and classical mechanics. (In the press.)

Fisher, R. A. 1937 The wave of advance of an advantageous gene. *Ann. Eugenics* **7**, 353–369.

Freidlin, M. 1985 *Functional integration and partial differential equations.* Annals of Mathematics Series, vol. 109. Princeton University Press.

Freidlin, M. 1991 Coupled reaction-diffusion equations. *Ann. Prob.* **19**, 29–57.

Friedman, M. J. & Doedel, E. 1991 Numerical computation and continuation of invariant manifolds connecting fixed points. *SIAM J. Numer. Analysis* **28**, 789–808.

Hartman, P. 1982 *Ordinary differential equations*, 2nd edn. Boston: Birkhäuser.

Karpelevich, F. I., Kelbert, M. Ya. & Suhov, Yu. M. 1993 On the asymptotical behaviour of a Markov branching random walk. (In preparation.)

Kolmogorov, A. N., Petrowski, I. & Piscounov, N. 1937 Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, *Mosc. Univ. Bull. Math.* **1**, 1–25.

McKean, H. P. 1975 Application of Brownian motion to the equation of Kolmogorov–Petrovskii–Piskunov. *Communs pure appl. Math.* **28**, 323–331.

McKean, H. P. 1976 Correction to the above. *Communs pure appl. Math.* **29**, 553–554.

Murray, J. D. 1989 *Mathematical biology.* Biomathematics Text, vol. 19. Heidelberg and New York: Springer-Verlag.

Neveu, J. 1987 Multiplicative martingales for spatial branching processes. *Seminar on stochastic processes* (ed. E. Çinlar, K. L Chung & R. K. Getoor). *Prog. Probab. Statist.* **15**, 223–241. Boston: Birkhäuser.

Seneta, E. 1981 *Non-negative matrices and Markov chains.* Heidelberg and New York: Springer-Verlag.

Uchiyama, K. 1981 Brownian first exit and sojourn over a one-sided moving boundary and applications. *Z. Wahrscheinlichkeitstheorie* **54**, 75–116.

Uchiyama, K. 1982 Spatial growth of a branching process of particles living in $\mathbb{R}^d$, *Ann. Probab.* **10**, 896–918.

Varadhan, S. R. S. 1984 *Large deviations and applications.* Philadelphia: SIAM.

Vol'pert, A. I. & Vol'pert, V. A. 1990 Application of the rotation theory of vector fields to the study of wave solutions of parabolic equations. *Transl. Trans. Moscow math. Soc.* **52**, 59–108.

Vol'pert, A. I. & Vol'pert, V. A. 1993 Travelling wave solutions of monotone parabolic systems. Université Claude Bernard, Lyon I. No. 146, CNRS URA 740.

Watanabe, S. 1967 Limit theorems for a class of branching processes. In *Markov processes and potential theory* (ed. J. Chover), pp. 205–232. New York: Wiley.

Williams, D. 1979 *Diffusions, Markov processes, and martingales*, vol. 1, *Foundations.* Chichester and New York: Wiley.

*Received 21 April 1993; accepted 6 August 1993*




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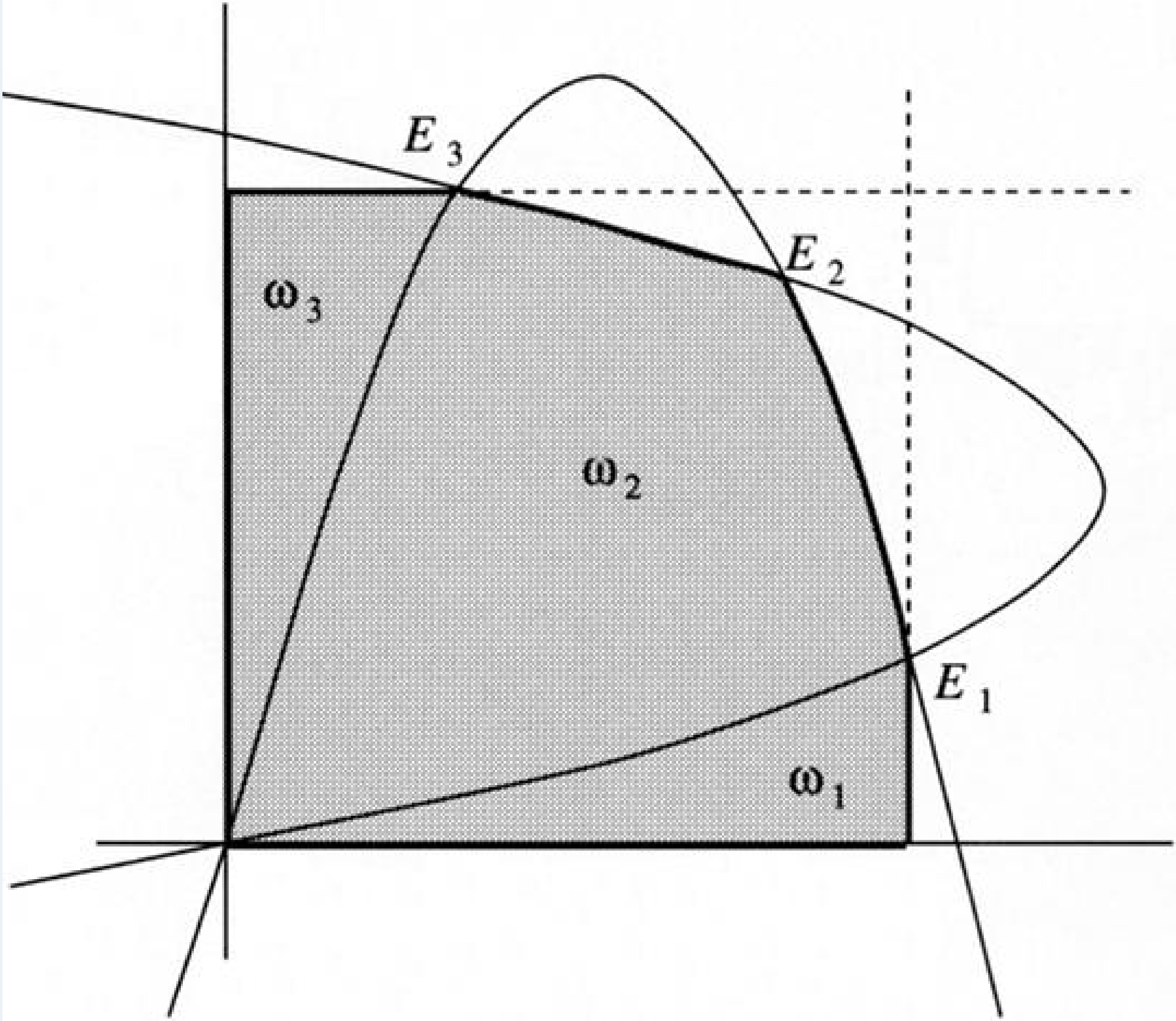


Figure 6.